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Approximation of the distribution of a stationary Markov process with application to option pricing

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We build a sequence of empirical measures on the space $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ of $\mathbb{R}^d$-valued cadlag functions on $\mathbb{R}_+$ in order to approximate the law of a stationary $\mathbb{R}^d$-valued Markov and Feller process (X_t). We obtain some general results on the convergence of this sequence. We then apply them to Brownian diffusions and solutions to Lévy-driven SDE's under some Lyapunov-type stability assumptions. As a numerical application of this work, we show that this procedure provides an efficient means of option pricing in stochastic volatility models.

Keywords: Euler scheme; Lévy process; numerical approximation; option pricing; stationary process; stochastic volatility model; tempered stable process

1. Introduction

1.1. Objectives and motivations

In this paper, we deal with an $\mathbb{R}^d$-valued Feller Markov process (X_t) with semigroup $(P_t)_{t\geq 0}$ and assume that (X_t) admits an invariant distribution ν_0. The aim of this work is to propose a way to approximate the whole stationary distribution $\mathbb{P}_{\nu_0}$ of (X_t). More precisely, we want to construct a sequence of weighted occupation measures $(\nu^{(n)}(\omega,\mathrm{d}\alpha))_{n\geq 1}$ on the Skorokhod space $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ such that $\nu^{(n)}(\omega,F)\stackrel{n\to+\infty}{\longrightarrow}\int F(\alpha)\mathbb{P}_{\nu_0}(\mathrm{d}\alpha)$ a.s. for a class of functionals $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ which includes bounded continuous functionals for the Skorokhod topology.

One of our motivations is to develop a new numerical method for option pricing in stationary stochastic volatility models which are slight modifications of the classical stochastic volatility models, where we suppose that the volatility evolves under its stationary regime.

1.2. Background and construction of the procedure

This work follows on from a series of recent papers due to Lamberton and Pagès ([12, 13]), Lemaire ([14, 15]) and Panloup ([18, 19, 20]), where the problem of the approximation of the invariant distribution is investigated for Brownian diffusions and for Lévy-driven SDE's.[1] In these papers, the algorithm is based on an adapted Euler scheme with decreasing step $(\gamma_k)_{k\geq 1}$. To be precise, let (Γ_n) be the sequence of discretization times: $\Gamma_0 = 0$, $\Gamma_n = \sum_{k=1}^n \gamma_k$ for every $n \geq 1$, and assume that $\Gamma_n \to +\infty$ when $n \to +\infty$. Let $(\bar{X}_{\Gamma_n})_{n\geq 0}$ be the Euler scheme obtained by "freezing" the coefficients between the Γ_n's and let $(\eta_n)_{n\geq 1}$ be a sequence of positive weights such that $H_n := \sum_{k=1}^n \eta_k \to +\infty$ when $k \to +\infty$. Then, under some Lyapunov-type stability assumptions adapted to the stochastic processes of interest, one shows that for a large class of steps and weights $(\eta_n, \gamma_n)_{n\geq 1}$,

$$\bar{\nu}_n(\omega, f) := \frac{1}{H_n} \sum_{k=1}^n \eta_k f(\bar{X}_{\Gamma_{k-1}}) \overset{n\to+\infty}{\longrightarrow} \int f(x)\nu_0(\mathrm{d}x) \qquad \text{a.s.,} \tag{1}$$

(at least)[2] for every bounded continuous function f.

Since the problem of the approximation of the invariant distribution has been deeply studied for a wide class of Markov processes (Brownian diffusions and Lévy-driven SDE's) and since the proof of (1) can be adapted to other classes of Markov processes under some specific Lyapunov assumptions, we choose in this paper to consider a general Markov process and to assume the existence of a time discretization scheme $(\bar{X}_{\Gamma_k})_{k\geq 0}$ such that (1) holds for the class of bounded continuous functions. The aim of this paper is then to investigate the convergence properties of a functional version of the sequence $(\bar{\nu}_n(\omega, \mathrm{d}\alpha))_{n\geq 1}$.

Let (X_t) be a Markov and Feller process and let $(\bar{X}_t)_{t\geq 0}$ be a stepwise constant time discretization scheme of (X_t) with non-increasing step sequence $(\gamma_n)_{n\geq 1}$ satisfying

$$\lim_{n\to+\infty} \gamma_n = 0, \qquad \Gamma_n := \sum_{k=1}^n \gamma_k \overset{n\to+\infty}{\longrightarrow} +\infty. \tag{2}$$

Letting $\Gamma_0 := 0$ and $\bar{X}_0 = x_0 \in \mathbb{R}^d$, we assume that

$$\bar{X}_t = \bar{X}_{\Gamma_n} \qquad \forall t \in [\Gamma_n, \Gamma_{n+1}[\tag{3}$$

and that $(\bar{X}_{\Gamma_n})_{n\geq 0}$ can be simulated recursively.

We denote by $(\mathcal{F}_t)_{t\geq 0}$ and $(\bar{\mathcal{F}}_t)_{t\geq 0}$ the usual augmentations of the natural filtrations $(\sigma(X_s, 0 \leq s \leq t))_{t\geq 0}$ and $(\sigma(\bar{X}_s, 0 \leq s \leq t))_{t\geq 0}$, respectively.

[1] Note that computing the invariant distribution is equivalent to computing the marginal laws of the stationary process (X_t) since $\nu_0 P_t = \nu_0$ for every $t \geq 0$.

[2] The class of functions for which (1) holds depends on the stability of the dynamical system. In particular, in the Brownian diffusion case, the convergence may hold for continuous functions with subexponential growth, whereas the class of functions strongly depends on the moments of the Lévy process when the stochastic process is a Lévy-driven SDE.

For $k \geq 0$, we denote by $(\bar{X}_t^{(k)})_{t\geq 0}$ the shifted process defined by

$$\bar{X}_t^{(k)} := \bar{X}_{\Gamma_k + t}.$$

In particular, $\bar{X}_t^{(0)} = \bar{X}_t$. We define a sequence of random probabilities $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ by

$$\nu^{(n)}(\omega, \mathrm{d}\alpha) = \frac{1}{H_n}\sum_{k=1}^{n} \eta_k \mathbf{1}_{\{\bar{X}^{(k-1)}(\omega)\in \mathrm{d}\alpha\}},$$

where $(\eta_k)_{k\geq 1}$ is a sequence of weights. For $t \geq 0$, $(\nu_t^{(n)}(\omega, \mathrm{d}x))_{n\geq 1}$ will denote the sequence of "marginal" empirical measures on $\mathbb{R}^d$ defined by

$$\nu_t^{(n)}(\omega, \mathrm{d}x) = \frac{1}{H_n}\sum_{k=1}^{n} \eta_k \mathbf{1}_{\{\bar{X}_t^{(k-1)}(\omega)\in \mathrm{d}x\}}.$$

1.3. Simulation of $(\nu^{(n)}(\omega, F))_{n\geq 1}$

For every functional $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, the following recurrence relation holds for every $n \geq 1$:

$$\nu^{(n+1)}(\omega, F) = \nu^{(n)}(\omega, F) + \frac{\eta_{n+1}}{H_{n+1}}(F(X^{(n)}(\omega)) - \nu^{(n)}(\omega, F)). \tag{4}$$

Then, if T is a positive number and $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$ is a functional depending only on the trajectory between 0 and T, $(\nu^{(n)}(\omega, F))_{n\geq 1}$ can be simulated by the following procedure.

Step 0. (i) Simulate $(\bar{X}_t^{(0)})_{t\geq 0}$ on $[0, T]$, that is, simulate $(\bar{X}_{\Gamma_k})_{k\geq 0}$ for $k = 0, \ldots, N(0, T)$, where

$$\begin{aligned} N(n, T) &:= \inf\{k \geq n, \Gamma_{k+1} - \Gamma_n > T\} \\ &= \max\{k \geq 0, \Gamma_k - \Gamma_n \leq T\}, \qquad n \geq 0, T > 0. \end{aligned} \tag{5}$$

Note that $n \mapsto N(n, t)$ is an increasing sequence since (γ_n) is non-increasing, and that

$$\Gamma_{N(n,T)} - \Gamma_n \leq T < \Gamma_{N(n,T)+1} - \Gamma_n.$$

(ii) Compute $F((\bar{X}_t^{(0)})_{t\geq 0})$ and $\nu^{(1)}(\omega, F)$. Store the values of $(\bar{X}_{\Gamma_k})$ for $k = 1, \ldots, N(0, T)$.

Step n ($n \geq 1$). (i) Since the values $(\bar{X}_{\Gamma_k})_{k\geq 0}$ are stored for $k = n, \ldots, N(n-1, T)$, simulate $(\bar{X}_{\Gamma_k})_{k\geq 0}$ for $k = N(n-1, T) + 1, \ldots, N(n, T)$ in order to obtain a path of $(\bar{X}_t^{(n)})$ on $[0, T]$.

(ii) Compute $F((\bar{X}_t^{(n)})_{t\geq 0})$ and use (4) to compute $\nu^{(n+1)}(\omega, F)$. Store the values of $(\bar{X}_{\Gamma_k})$ for $k = n + 1, \ldots, N(n, T)$.

Remark 1. As shown in the description of the procedure, one generally has to store the vector $[\bar{X}_{\Gamma_n}, \ldots, \bar{X}_{\Gamma_{N(n,T)}}]$ at time n. Since (γ_n) is a sequence with infinite sum that decreases to 0, it follows that the size of this vector increases "slowly" to $+\infty$. For instance, if $\gamma_n = Cn^{-\rho}$ with $\rho \in (0,1)$, its size is of order n^ρ. However, it is important to remark that even though the number of values to be stored tends to $+\infty$, that is not always the case for the number of operations at each step. Indeed, since $\bar{X}^{(n+1)}$ is obtained by shifting $\bar{X}^{(n)}$, it is usually possible to use, at step $n+1$, the preceding computations and to simulate the sequence $(F(\bar{X}^{(n)}))_{n\geq 0}$ in a "quasi-recursive" way. For instance, such remark holds for Asian options because the associated pay-off can be expressed as a function of an additive functional (see Section 5 for simulations).

Before outlining the sequel of the paper, we list some notation linked to the spaces $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $\mathbb{D}([0,T], \mathbb{R}^d)$ of cadlag $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$ and $[0,T]$, respectively, endowed with the Skorokhod topology. First, we denote by d_1 the Skorokhod distance on $\mathbb{D}([0,1], \mathbb{R}^d)$ defined for every α, $\beta \in \mathbb{D}([0,1], \mathbb{R}^d)$ by

$$d_1(\alpha, \beta) = \inf_{\lambda \in \Lambda_1} \left\{ \max\left(\sup_{t\in[0,1]} |\alpha(t) - \beta(\lambda(t))|, \sup_{0\leq s<t\leq 1} \left| \log\left(\frac{\lambda(t) - \lambda(s)}{t-s} \right) \right| \right) \right\},$$

where Λ_1 denotes the set of increasing homeomorphisms of $[0,1]$. Second, for $T>0$, $\phi_T : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \mapsto \mathbb{D}([0,1], \mathbb{R}^d)$ is the function defined by $(\phi_T(\alpha))(s) = \alpha(sT)$ for every $s \in [0,1]$. We then denote by d the distance on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ defined for every $\alpha, \beta \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ by

$$d(\alpha, \beta) = \int_0^{+\infty} \mathrm{e}^{-t} (1 \wedge d_1(\phi_t(\alpha), \phi_t(\beta)))\, \mathrm{d}t. \tag{6}$$

We recall that $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d), d)$ is a Polish space and that the induced topology is the usual Skorokhod topology on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ (see, e.g., Pagès [16]). For every $T>0$, we set

$$\mathcal{D}_T = \bigcap_{s>T} \sigma(\pi_u, 0 \leq u \leq s),$$

where $\pi_s : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d$ is defined by $\pi_s(\alpha) = \alpha(s)$. For a functional $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, F_T denotes the functional defined for every $\alpha \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ by

$$F_T(\alpha) = F(\alpha^T) \qquad \text{with } \alpha^T(t) = \alpha(t \wedge T)\ \forall t \geq 0. \tag{7}$$

Finally, we will say that a functional $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$ is Sk-continuous if F is continuous for the Skorokhod topology on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and the notation "$\overset{(Sk)}{\Longrightarrow}$" will denote the weak convergence on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

In Section 2, we state our main results for a general $\mathbb{R}^d$-valued Feller Markov process. Then, in Section 3, we apply them to Brownian diffusions and Lévy-driven SDE's. Section 4 is devoted to the proofs of the main general results. Finally, in Section 5, we complete this paper with an application to option pricing in stationary stochastic volatility models.

2. General results

In this section, we state the results on convergence of the sequence $(\nu^{(n)}(\omega, d\alpha))_{n\geq 1}$ when (X_t) is a general Feller Markov process.

2.1. Weak convergence to the stationary regime

As explained in the Introduction, since the a.s. convergence of $(\nu_0^{(n)}(\omega, dx))_{n\geq 1}$ to the invariant distribution ν_0 has already been deeply studied for a large class of Markov processes (Brownian diffusions and Lévy driven SDE's), our approach will be to derive the convergence of $(\nu^{(n)}(\omega, d\alpha))_{n\geq 1}$ toward $\mathbb{P}_{\nu_0}$ from that of $(\nu_0^{(n)}(\omega, dx))_{n\geq 1}$ to the invariant distribution ν_0. More precisely, we will assume in Theorem 1 that

($\mathbf{C_{0,1}}$): (X_t) admits a unique invariant distribution ν_0 and

$$\nu_0^{(n)}(\omega, dx) \overset{n\to+\infty}{\Longrightarrow} \nu_0(dx) \qquad \text{a.s.},$$

whereas in Theorem 2, we will only assume that

($\mathbf{C_{0,2}}$): $(\nu_0^{(n)}(\omega, dx))_{n\geq 1}$ is a.s. tight on $\mathbb{R}^d$.

We also introduce three other assumptions, ($\mathbf{C_1}$), ($\mathbf{C_2}$) and ($\mathbf{C}_{\mathbf{3},\varepsilon}$), regarding the continuity in probability of the flow $x \mapsto (X_t^x)$, the asymptotic convergence of the shifted time discretization scheme to the true process (X_t) and the steps and weights, respectively.

($\mathbf{C_1}$): For every $x_0 \in \mathbb{R}^d$, $\epsilon > 0$ and $T > 0$,

$$\limsup_{x_0\to x} \mathbb{P}\left(\sup_{0\leq t\leq T} |X_t^x - X_t^{x_0}| \geq \epsilon\right) = 0. \tag{8}$$

($\mathbf{C_2}$): $(\bar{X}_t)$ is a non-homogeneous Markov process and for every $n \geq 0$, it is possible to construct a family of stochastic processes $(Y_t^{(n,x)})_{x\in\mathbb{R}^d}$ such that

(i) $\mathcal{L}(Y^{(n,x)}) \overset{\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)}{=} \mathcal{L}(\bar{X}^{(n)} | \bar{X}_0^{(n)} = x)$;

(ii) for every compact set K of $\mathbb{R}^d$, for every $T \geq 0$,

$$\sup_{x\in K} \sup_{0\leq t\leq T} |Y_t^{(n,x)} - X_t^x| \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{in probability.} \tag{9}$$

($\mathbf{C}_{\mathbf{3},\varepsilon}$): For every $n \geq 1$, $\eta_n \leq C\gamma_n H_n^{\varepsilon}$.

Remark 2. Assumption ($\mathbf{C_2}$) implies, in particular, that asymptotically and uniformly on compact sets of $\mathbb{R}^d$, the law of the approximate process $(\bar{X}^{(n)})$, given its initial value, is close to that of the true process.

If there exists a unique invariant distribution ν_0, the second part of ($\mathbf{C_2}$) can be relaxed to the following, less stringent, assertion: for all $\epsilon > 0$, there exists a compact set $A_\epsilon \subset \mathbb{R}^d$ such that $\nu_0(A_\epsilon^c) \leq \epsilon$ and such that

$$\sup_{x\in A_\epsilon} \sup_{0\leq t\leq T} |Y_t^{(n,x)} - X_t^x| \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{in probability.} \tag{10}$$

This weaker assumption can some times be needed in stochastic volatility models like the Heston model (see Section 5 for details).

The preceding assumptions are all that we require for the convergence of $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ to $\mathbb{P}_{\nu_0}$ along the bounded Sk-continuous functionals, that is, for the a.s. weak convergence on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. However, the integration of non-bounded continuous functionals $F : \mathbb{D}([0,T], \mathbb{R}^d) \to \mathbb{R}$ will need some additional assumptions, depending on the stability of the time discretization scheme and on the steps and weights sequences. We will suppose that F is dominated (in a sense to be specified later) by a function $\mathcal{V} : \mathbb{R}^d \to \mathbb{R}_+$ that satisfies the following assumptions for some $s \geq 2$ and $\varepsilon < 1$.

$\mathbf{H}(\mathbf{s}, \varepsilon)$: For every $T > 0$,

(i) $\sup_{n\geq 1} \mathbb{E}\left[\sup_{0\leq t\leq T} \mathcal{V}^s(Y_t^{(n,x)})\right] \leq C_T \mathcal{V}^s(x)$,

(ii) $\sup_{n\geq 1} \nu_0^{(n)}(\mathcal{V}) < +\infty$,

(iii) $\sum_{k\geq 1} \frac{\eta_k}{H_k^2} \mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_{k-1}})] < +\infty$,

(iv) $\sum_{k\geq 1} \frac{\Delta N(k,T)}{H_k^s} \mathbb{E}[\mathcal{V}^{s(1-\varepsilon)}(\bar{X}_{\Gamma_{k-1}})] < +\infty$,

where $T \mapsto C_T$ is locally bounded on $\mathbb{R}_+$ and $\Delta N(k,T) = N(k,T) - N(k-1,T)$.

For every $\varepsilon < 1$, we then set

$$\mathcal{K}(\varepsilon) = \{\mathcal{V} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}_+), \mathbf{H}(\mathbf{s}, \varepsilon) \text{ holds for some } s \geq 2\}.$$

Remark 3. Apart from assumption (i), which is a classical condition on the finite time horizon control, the assumptions in $\mathbf{H}(\mathbf{s}, \varepsilon)$ strongly rely on the stability of the time discretization scheme (and then, to that of the true process). More precisely, we will see when we apply our general results to SDE's that these properties are some consequences of the Lyapunov assumptions needed for the tightness of $(\nu_0^{(n)}(\omega, \mathrm{d}x))_{n\geq 1}$.

We can now state our first main result.

Theorem 1. *Assume* $(\mathbf{C_{0,1}})$, $(\mathbf{C_1})$, $(\mathbf{C_2})$ *and* $(\mathbf{C}_{\mathbf{3},\varepsilon})$ *with* $\varepsilon \in (-\infty, 1)$. *Then, a.s., for every bounded* Sk*-continuous functional* $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$,

$$\nu^{(n)}(\omega, F) \overset{n\to+\infty}{\longrightarrow} \int F(\alpha) \mathbb{P}_{\nu_0}(\mathrm{d}\alpha), \tag{11}$$

where $\mathbb{P}_{\nu_0}$ *denotes the stationary distribution of* (X_t) *(with initial law* ν_0*).*

Furthermore, for every $T > 0$, *for every non-bounded* Sk*-continuous functional* $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, *(11) holds a.s. for* F_T *(defined by (7)) if there exists* $\mathcal{V} \in \mathcal{K}(\varepsilon)$ *and*

$\rho \in [0,1)$ *such that*

$$|F_T(\alpha)| \leq C \sup_{0\leq t\leq T} \mathcal{V}^\rho(\alpha_t) \qquad \forall \alpha \in \mathbb{D}(\mathbb{R}_+,\mathbb{R}^d). \tag{12}$$

In the second result, the uniqueness of the invariant distribution is not required and the sequence $(\nu_0^{(n)}(\omega,\mathrm{d}x))_{n\geq 1}$ is only supposed to be tight.

Theorem 2. *Assume* $(\mathbf{C_{0,2}})$, $(\mathbf{C_1})$, $(\mathbf{C_2})$ *and* $(\mathbf{C_{3,\varepsilon}})$ *with* $\varepsilon \in (-\infty,1)$. *Assume that* $(\nu_0^{(n)}(\omega,\mathrm{d}x))_{n\geq 1}$ *is a.s. tight on* $\mathbb{R}^d$. *We then have the following.*

(i) *The sequence* $(\nu^{(n)}(\omega,\mathrm{d}\alpha))_{n\geq 1}$ *is a.s. tight on* $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ *and a.s., for every convergent subsequence* $(n_k(\omega))_{n\geq 1}$, *for every bounded* Sk*-continuous functional* $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$,

$$\nu^{(n_k(\omega))}(\omega,F) \overset{n\to+\infty}{\longrightarrow} \int F(\alpha)\mathbb{P}_{\nu_\infty}(\mathrm{d}\alpha), \tag{13}$$

where $\mathbb{P}_{\nu_\infty}$ *is the law of* (X_t) *with initial law* ν_∞ *being a weak limits for* $(\nu_0^{(n)}(\omega,\mathrm{d}x))_{n\geq 1}$.

Furthermore, for every $T>0$, *for every non-bounded* Sk*-continuous functional* $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$, *(13) holds a.s. for* F_T *if (12) is satisfied with* $\mathcal{V}\in\mathcal{K}(\varepsilon)$ *and* $\rho\in[0,1)$.

(ii) *If, moreover,*

$$\frac{1}{H_n}\sum_{k=1}^{n} \max_{l\geq k+1} \frac{|\Delta\eta_\ell|}{\gamma_\ell} \overset{n\to+\infty}{\longrightarrow} 0, \tag{14}$$

then ν_∞ *is necessarily an invariant distribution for the Markov process* (X_t).

Remark 4. Condition (14) holds for a large class of steps and weights. For instance, if $\eta_n = C_1 n^{-\rho_1}$ and $\gamma_n = C_2 n^{-\rho_2}$ with $\rho_1\in[0,1]$ and $\rho_2\in(0,1]$, then (14) is satisfied if $\rho_1 = 0$ or if $\rho_1 \in (\max(0,2\rho_2-1),1)$.

2.2. Extension to the non-stationary case

Even though the main interest of this algorithm is the weak approximation of the process when stationary, we observe that when ν_0 is known, the algorithm can be used to approximate $\mathbb{P}_{\mu_0}$ if μ_0 is a probability on $\mathbb{R}^d$ that is absolutely continuous with respect to ν_0.

Indeed, assume that $\mu_0(\mathrm{d}x) = \phi(x)\nu_0(\mathrm{d}x)$, where $\phi:\mathbb{R}^d\to\mathbb{R}$ is a continuous nonnegative function. For a functional $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$, denote by F^ϕ the functional defined on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ by $F^\phi(\alpha) = F(\alpha)\phi(\alpha(0))$.

Then, if $\nu^{(n)}(\omega,\mathrm{d}\alpha) \overset{(Sk)}{\Rightarrow} \mathbb{P}_{\nu_0}(\mathrm{d}\alpha)$ a.s., we also have the following convergence: a.s., for every bounded Sk-continuous functional $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$,

$$\nu^{(n)}(\omega,F^\phi) \overset{n\to+\infty}{\longrightarrow} \int F^\phi(\alpha)\mathbb{P}_{\nu_0}(\mathrm{d}\alpha) = \int F(\alpha)\mathbb{P}_{\mu_0}(\mathrm{d}\alpha).$$

3. Application to Brownian diffusions and Lévy-driven SDE's

Let $(X_t)_{t\geq 0}$ be a cadlag stochastic process solution to the SDE

$$\mathrm{d}X_t = b(X_{t^-})\,\mathrm{d}t + \sigma(X_{t^-})\,\mathrm{d}W_t + \kappa(X_{t^-})\,\mathrm{d}Z_t, \tag{15}$$

where $b:\mathbb{R}^d \to \mathbb{R}^d$, $\sigma:\mathbb{R}^d \mapsto \mathbb{M}_{d,\ell}$ (set of $d\times\ell$ real matrices) and $\kappa:\mathbb{R}^d \mapsto \mathbb{M}_{d,\ell}$ are continuous functions with sublinear growth, $(W_t)_{t\geq 0}$ is an ℓ-dimensional Brownian motion and $(Z_t)_{t\geq 0}$ is an integrable purely discontinuous $\mathbb{R}^\ell$-valued Lévy process independent of $(W_t)_{t\geq 0}$ with Lévy measure π and characteristic function given for every $t\geq 0$ by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle u, Z_t\rangle}] = \exp\left[t\left(\int \mathrm{e}^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u,y\rangle \pi(\mathrm{d}y)\right)\right].$$

Let $(\gamma_n)_{n\geq 1}$ be a non-increasing step sequence satisfying (2). Let $(U_n)_{n\geq 1}$ be a sequence of i.i.d. random variables such that $U_1 \stackrel{\mathcal{L}}{=} \mathcal{N}(0, I_\ell)$ and let $\xi := (\xi_n)_{n\geq 1}$ be a sequence of independent $\mathbb{R}^\ell$-valued random variables, independent of $(U_n)_{n\geq 1}$. We then denote by $(\bar{X}_t)_{t\geq 0}$ the stepwise constant Euler scheme of (X_t) for which $(\bar{X}_{\Gamma_n})_{n\geq 0}$ is recursively defined by $\bar{X}_0 = x \in \mathbb{R}^d$ and

$$\bar{X}_{\Gamma_{n+1}} = \bar{X}_{\Gamma_n} + \gamma_{n+1} b(\bar{X}_{\Gamma_n}) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_{\Gamma_n})U_{n+1} + \kappa(\bar{X}_{\Gamma_n})\xi_{n+1}. \tag{16}$$

We recall that the increments of (Z_t) cannot be simulated in general. That is why we generally need to construct the sequence (ξ_n) with some approximations of the true increments. We will come back to this construction in Section 3.2.

As in the general case, we denote by $(\bar{X}^{(k)})_{k\geq 0}$ and $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ the sequences of associated shifted Euler schemes and empirical measures, respectively.

Let us now introduce some Lyapunov assumptions for the SDE. Let $\mathcal{EQ}(\mathbb{R}^d)$ denote the set of *essentially quadratic* $\mathcal{C}^2$-functions $V:\mathbb{R}^d \to \mathbb{R}_+^*$ such that $\lim V(x) = +\infty$ as $|x| \to +\infty$, $|\nabla V| \leq C\sqrt{V}$ and D^2V is bounded. Let $a\in(0,1]$ denote the mean reversion intensity. The Lyapunov (or mean reversion) assumption is the following.

$(\mathbf{S_a})$: There exists a function $V \in \mathcal{EQ}(\mathbb{R}^d)$ such that:

(i) $|b|^2 \leq CV^a$, $\mathrm{Tr}(\sigma\sigma^*(x)) + \|\kappa(x)\|^2 \stackrel{|x|\to+\infty}{=} o(V^a(x))$;
(ii) there exist $\beta\in\mathbb{R}$ and $\rho > 0$ such that $\langle \nabla V, b\rangle \leq \beta - \rho V^a$.

From now on, we separate the Brownian diffusions and Lévy-driven SDE cases.

3.1. Application to Brownian diffusions

In this part, we assume that $\kappa = 0$. We recall a result by Lamberton and Pagès [13].

Proposition 1. *Let $a\in(0,1]$ such that $(\mathbf{S_a})$ holds. Assume that the sequence $(\eta_n/\gamma_n)_{n\geq 1}$ is non-increasing.*

(a) *Let* $(\theta_n)_{n\geq 1}$ *be a sequence of positive numbers such that* $\sum_{n\geq 1}\theta_n\gamma_n < +\infty$ *and that there exists* $n_0\in\mathbb{N}$ *such that* $(\theta_n)_{n\geq n_0}$ *is non-increasing. Then, for every positive* r,

$$\sum_{n\geq 1}\theta_n\gamma_n\mathbb{E}[V^r(\bar{X}_{\Gamma_{n-1}})] < +\infty.$$

(b) *For every* $r>0$,

$$\sup_{n\geq 1}\nu_0^{(n)}(\omega, V^r) < +\infty \qquad a.s. \tag{17}$$

Hence, the sequence $(\nu_0^{(n)}(\omega,\mathrm{d}x))_{n\geq 1}$ *is a.s. tight.*

(c) *Moreover, every weak limit of this sequence is an invariant probability for the SDE (15). In particular, if* $(X_t)_{t\geq 0}$ *admits a unique invariant probability* ν_0, *then for every continuous function* f *such that* $f\leq CV^r$ *with* $r>0$, $\lim_{n\to\infty}\nu_0^{(n)}(\omega,f)=\nu_0(f)$ *a.s.*

Remark 5. For instance, if $V(x)=1+|x|^2$, then the preceding convergence holds for every continuous function with polynomial growth. According to Theorem 3.2 in Lemaire [14], it is possible to extend these results to continuous functions with exponential growth, but it then strongly depends on σ. Further the conditions on steps and weights can be less restrictive and may contain the case $\eta_n=1$, for instance (see Remark 4 of Lamberton and Pagès [13] and Lemaire [14]).

We then derive the following result from the preceding proposition and from Theorems 1 and 2.

Theorem 3. *Assume that* b *and* σ *are locally Lipschitz functions and that* $\kappa=0$. *Let* $a\in(0,1]$ *such that* $(\mathbf{S_a})$ *holds and assume that* (η_n/γ_n) *is non-increasing.*

(a) *The sequence* $(\nu^{(n)}(\omega,\mathrm{d}\alpha))_{n\geq 1}$ *is a.s. tight on* $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$[3] *and every weak limit of* $(\nu^{(n)}(\omega,\mathrm{d}\alpha))_{n\geq 1}$ *is the distribution of a stationary process solution to (15). In particular, when uniqueness holds for the invariant distribution* ν_0, *a.s., for every bounded continuous functional* $F:\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$,

$$\nu^{(n)}(\omega,F) \overset{n\to+\infty}{\longrightarrow} \int F(x)\mathbb{P}_{\nu_0}(\mathrm{d}x). \tag{18}$$

(b) *Furthermore, if there exists* $s\in(2,+\infty)$ *and* $n_0\in\mathbb{N}$ *such that*

$$\left(\frac{\Delta N(k,T)}{\gamma_k H_k^s}\right)_{n\geq n_0} \textit{is non-increasing and} \sum_{k\geq 1}\frac{\Delta N(k,T)}{H_k^s} < +\infty, \tag{19}$$

[3] $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}_+$ with values in $\mathbb{R}^d$ endowed with the topology of uniform convergence on compact sets.

then, for every $T>0$, for every non-bounded continuous functional $F:\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$, (18) holds for F_T if the following condition is satisfied:

$$\exists r>0 \quad \textit{such that} \quad |F_T(\alpha)|\le C\sup_{0\le t\le T} V^r(\alpha_t) \qquad \forall \alpha\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d).$$

Remark 6. If $\eta_n=C_1n^{-\rho_1}$ and $\gamma_n=C_2n^{-\rho_2}$ with $0<\rho_2\le\rho_1\le 1$, then for $s\in(1,+\infty)$, (19) is fulfilled if and only if $s>1/(1-\rho_1)$. It follows that there exists $s\in(2,+\infty)$ such that (19) holds as soon as $\rho_1<1$.

Proof of Theorem 3. We want to apply Theorem 2. First, by Proposition 1, assumption $(\mathbf{C_{0,2}})$ is fulfilled and every weak limit of $(\nu_0^{(n)}(\omega,\mathrm{d}x))$ is an invariant distribution. Second, it is well known that $(\mathbf{C_1})$ and $(\mathbf{C_2})$ are fulfilled when b and σ are locally Lispchitz sublinear functions. Then, since $(\mathbf{C}_{\mathbf{3},\varepsilon})$ holds with $\varepsilon=0$, (18) holds for every bounded continuous functional F. Finally, one checks that $\mathbf{H}(\mathbf{s},\mathbf{0})$ holds with $\mathcal{V}:=V^r$ $(r>0)$. It is classical that assumption (a) is true when b and σ are sublinear. Assumption (b) follows from Proposition 1(b). Let $\theta_{n,1}=\eta_n/(\gamma_nH_n^2)$ and $\theta_{n,2}=\Delta N(n,T)/(\gamma_nH_n^s)$. Using (19) and the fact that (η_n/γ_n) is non-increasing yields that $(\theta_{n,1})$ and $(\theta_{n,2})$ satisfy the conditions of Proposition 1 (see (35) for details). Then, (iii) and (iv) of $\mathbf{H}(\mathbf{s},\mathbf{0})$ are consequences of Proposition 1(a). This completes the proof. □

3.2. Application to Lévy-driven SDE's

When we want to extend the results obtained for Brownian SDE's to Lévy-driven SDE's, one of the main difficulties comes from the moments of the jump component (see Panloup [18] for details). For simplification, we assume here that (Z_t) has a moment of order $2p\ge 2$, that is, that its Lévy measure π satisfies the following assumption with $p\ge 1$:

$$(\mathbf{H_p^1}):\int_{|y|>1}\pi(\mathrm{d}y)|y|^{2p}<+\infty.$$

We also introduce an assumption about the behavior of the moments of the Lévy measure at 0:

$$(\mathbf{H_q^2}):\int_{|y|\le 1}\pi(\mathrm{d}y)|y|^{2q}<+\infty, \qquad q\in[0,1].$$

This assumption ensures that (Z_t) has finite $2q$-variations. Since $\int_{|y|\le 1}|y|^2\pi(\mathrm{d}y)$ is finite, this is always satisfied for $q=1$.

Let us now specify the law of (ξ_n) introduced in (16). When the increments of (Z_t) can be exactly simulated, we denote by (E) the Euler scheme and by $(\xi_{n,E})$ the associated sequence

$$\xi_{n,E}\overset{\mathcal{L}}{=}Z_{\gamma_n} \qquad \forall n\ge 1.$$

When the increments of (Z_t) cannot be simulated, we introduce some *approximated* Euler schemes (P) and (W) built with some sequences $(\xi_{n,P})$ and $(\xi_{n,W})$ of approximations of the true increment (see Panloup [19] for more detailed presentations of these schemes).

In scheme (P),

$$\xi_{n,P} \stackrel{\mathcal{L}}{=} Z_{\gamma_n,n},$$

where $(Z_{\cdot,n})_{n\geq 1}$ a sequence of compensated compound Poisson processes obtained by truncating the small jumps of $(Z_t)_{t\geq 0}$:

$$Z_{t,n} := \sum_{0<s\leq t} \Delta Z_s 1_{\{|\Delta Z_s|>u_n\}} - t\int_{|y|>u_n} y\pi(\mathrm{d}y) \qquad \forall t \geq 0, \tag{20}$$

where $(u_n)_{n\geq 1}$ is a sequence of positive numbers such that $u_n \to 0$. We recall that $Z_{\cdot,n} \stackrel{n\to+\infty}{\longrightarrow} Z$ locally uniformly in L^2 (see, e.g., Protter [21]).

As shown in Panloup [19], the error induced by this approximation is very large when the local behavior of the small jumps component is irregular. However, it is possible to refine this approximation by a *Wienerization* of the small jumps, that is, by replacing the small jumps by a linear transform of a Brownian motion instead of discarding them (see Asmussen and Rosinski [2]). The corresponding scheme is denoted by (W) with $\xi_{n,W}$ satisfying

$$\xi_{n,W} \stackrel{\mathcal{L}}{=} \xi_{n,P} + \sqrt{\gamma_n} Q_n \Lambda_n \qquad \forall n \geq 1,$$

where $(\Lambda_n)_{n\geq 1}$ is a sequence of i.i.d. random variables, independent of $(\xi_{n,P})_{n\geq 1}$ and $(U_n)_{n\geq 1}$, such that $\Lambda_1 \stackrel{\mathcal{L}}{=} \mathcal{N}(0, I_\ell)$ and (Q_n) is a sequence of $\ell \times \ell$ matrices such that

$$(Q_n Q_n^*)_{i,j} = \int_{|y|\leq u_k} y_i y_j \pi(\mathrm{d}y).$$

We recall the following result obtained in Panloup [18] in our slightly simplified framework.

Proposition 2. *Let $a \in (0,1]$, $p \geq 1$ and $q \in [0,1]$ such that $(\mathbf{H^1_p})$, $(\mathbf{H^2_q})$ and $(\mathbf{S_a})$ hold. Assume that the sequence $(\eta_n/\gamma_n)_{n\geq 1}$ is non-increasing. Then, the following assertions hold for schemes* (E)*,* (P) *and* (W)*.*

(a) *Let (θ_n) satisfy the conditions of Proposition 1. Then, $\sum_{n\geq 1} \theta_n \gamma_n \mathbb{E}[V^{p+a-1}(\bar{X}_{\Gamma_{n-1}})] < +\infty$.*

(b) *We have*

$$\sup_{n\geq 1} \nu_0^{(n)}(\omega, V^{p/2+a-1}) < +\infty \qquad a.s. \tag{21}$$

Hence, the sequence $(\nu_0^{(n)}(\omega, \mathrm{d}x))_{n\geq 1}$ is a.s. tight as soon as $p/2 + a - 1 > 0$.

(c) *Moreover, if* $\mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^{2q} \le CV^{p/2+a-1}$, *then every weak limit of this sequence is an invariant probability for the SDE (15). In particular, if* $(X_t)_{t\ge0}$ *admits a unique invariant probability* ν_0, *for every continuous function* f *such that* $f = o(V^{p/2+a-1})$, $\lim_{n\to\infty} \nu_0^{(n)}(\omega, f) = \nu_0(f)$ *a.s.*

Remark 7. For schemes (E) and (P), the above proposition is a direct consequence of Theorem 2 and Proposition 2 of Panloup [18]. As concerns scheme (W), a straightforward adaptation of the proof yields the result.

Our main functional result for Lévy-driven SDE's is then the following.

Theorem 4. *Let* $a \in (0,1]$ *and* $p \ge 1$ *such that* $p/2 + a - 1 > 0$ *and let* $q \in [0,1]$. *Assume* $(\mathbf{H_p^1})$, $(\mathbf{H_q^2})$ *and* $(\mathbf{S_a})$. *Assume that* b, σ *and* κ *are locally Lipschitz functions. If, moreover,* $(\eta_n/\gamma_n)_{n\ge1}$ *is non-increasing, then the following result holds for schemes* (E), (P) *and* (W).

(a) *The sequence* $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\ge1}$ *is a.s. tight on* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. *Moreover, if*

$$\mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^{2q} \le CV^{p/2+a-1} \quad \text{or} \quad \frac{1}{H_n}\sum_{k=1}^{n} \max_{l\ge k+1} \frac{|\Delta\eta_\ell|}{\gamma_{\ell-1}} \overset{n\to+\infty}{\longrightarrow} 0, \tag{22}$$

then every weak limit of $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\ge1}$ *is the distribution of a stationary process solution to (15).*

(b) *Assume that the invariant distribution is unique. Let* $\varepsilon \le 0$ *such that* $(\mathbf{C}_{\mathbf{3},\varepsilon})$ *holds. Then, a.s., for every* $T > 0$, *for every Sk-continuous functional* $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, *(18) holds for* F_T *if there exist* $\rho \in [0,1)$ *and* $s \ge 2$, *such that*

$$|F_T(\alpha)| \le C \sup_{0\le t\le T} V^{(\rho(p+a-1))/s}(\alpha_t) \qquad \forall \alpha \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$$

and if

$$\left(\frac{\Delta N(k,T)}{\gamma_k H_k^{s(1-\varepsilon)}}\right)_{n\ge n_0} \text{ is non-increasing and } \sum_{k\ge1} \frac{\Delta N(k,T)}{H_k^{s(1-\varepsilon)}} < +\infty. \tag{23}$$

Remark 8. In (22), both assumptions imply the invariance of every weak limit of $(\nu_0^{(n)}(\omega, \mathrm{d}x))$. These two assumptions are very different. The first is needed in Proposition 2 for using the Echeverria–Weiss invariance criteria (see Ethier and Kurtz [7], page 238, Lamberton and Pagès [12] and Lemaire [14]), whereas the second appears in Theorem 2, where our functional approach shows that under some mild additional conditions on steps and weights, every weak limit is always invariant.

For (23), we refer to Remark 6 for simple sufficient conditions when (γ_n) and (η_n) are some polynomial steps and weights.

4. Proofs of Theorems 1 and 2

We begin the proof with some technical lemmas. In Lemma 1, we show that the $a.s$ weak convergence of the random measures $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ can be characterized by the convergence (11) along the set of bounded Lipschitz functionals F for the distance d. Then, in Lemma 2, we show with some martingale arguments that if the functional F depends only on the restriction of the trajectory to $[0,T]$, then the convergence of $(\nu^{(n)}(\omega, F))_{n\geq 1}$ is equivalent to that of a more regular sequence. This step is fundamental for the sequel of the proof.

Finally, Lemma 4 is needed for the proof of Theorem 2. We show that under some mild conditions on the step and weight sequences, any Markovian weak limit of the sequence $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ is stationary.

4.1. Preliminary lemmas

Lemma 1. *Let (E,d) be a Polish space and let $\mathcal{P}(E)$ denote the set of probability measures on the Borel σ-field $\mathcal{B}(E)$, endowed with the weak convergence topology. Let $(\mu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ be a sequence of random probabilities defined on $\Omega \times \mathcal{B}(E)$.*

(a) *Assume that there exists $\mu^{(\infty)} \in \mathcal{P}(E)$ such that for every bounded Lipschitz function $F : E \to \mathbb{R}$,*

$$\mu^{(n)}(\omega, F) \overset{n\to+\infty}{\longrightarrow} \mu^{(\infty)}(F) \qquad a.s. \tag{24}$$

Then, a.s., $(\mu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ converges weakly to $\mu^{(\infty)}$ on $\mathcal{P}(E)$.

(b) *Let $\mathcal{U}$ be a subset of $\mathcal{P}(E)$. Assume that for every sequence $(F_k)_{k\geq 1}$ of Lipschitz and bounded functions, a.s., for every subsequence $(\mu^{(\phi_\omega(n))}(\omega, \mathrm{d}\alpha))$, there exists a subsequence $(\mu^{(\phi_\omega\circ\psi_\omega(n))}(\omega, \mathrm{d}\alpha))$ and a $\mathcal{U}$-valued random probability $\mu^{(\infty)}(\omega, \mathrm{d}\alpha)$ such that for every $k \geq 1$,*

$$\mu^{(\psi_\omega\circ\phi_\omega(n))}(\omega, F_k) \overset{n\to+\infty}{\longrightarrow} \mu^{(\infty)}(\omega, F_k) \qquad a.s. \tag{25}$$

Then, $(\mu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ is a.s. tight with weak limits in $\mathcal{U}$.

Proof. We do not give a detailed proof of the next lemma, which is essentially based on the fact that in a separable metric space (E,d), one can build a sequence of bounded Lipschitz functions $(g_k)_{k\geq 1}$ such that for any sequence $(\mu_n)_{n\geq 1}$ of probability measures on $\mathcal{B}(E)$, $(\mu_n)_{n\geq 1}$ weakly converges to a probability μ if and only if the convergence holds along the functions g_k, $k \geq 1$ (see Parthasarathy [22], Theorem 6.6, page 47 for a very similar result). □

For every $n \geq 0$, for every $T > 0$, we introduce $\tau(n,T)$ defined by

$$\tau(n,T) := \min\{k \geq 0, N(k,T) \geq n\} = \min\{k \leq n, \Gamma_k + T \geq \Gamma_n\}. \tag{26}$$

Note that for $k \in \{0, \ldots, \tau(n,T)-1\}$, $\{\bar{X}_t^{(k)}, 0 \le t \le T\}$ is $\bar{\mathcal{F}}_{\Gamma_n}$-measurable and

$$T - \gamma_{\tau(n,T)-1} \le \Gamma_n - \Gamma_{\tau(n,T)} \le T.$$

Lemma 2. *Assume* $(\mathbf{C}_{\mathbf{3},\varepsilon})$ *with* $\varepsilon < 1$. *Let* $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$ *be a Sk-continuous functional. Let* $(\mathcal{G}_k)$ *be a filtration such that* $\bar{\mathcal{F}}_{\Gamma_k} \subset \mathcal{G}_k$ *for every* $k \ge 1$. *Then, for any* $T > 0$:

(a) *if* F_T *(defined by (7)) is bounded,*

$$\frac{1}{H_n} \sum_{k=1}^{n} \eta_k \big(F_T(\bar{X}^{(k-1)}) - \mathbb{E}[F_T(\bar{X}^{(k-1)})/\mathcal{G}_{k-1}]\big) \overset{n\to+\infty}{\longrightarrow} 0 \qquad a.s.; \tag{27}$$

(b) *if* F_T *is not bounded, (27) holds if there exists* $\mathcal{V} : \mathbb{R}^d \to \mathbb{R}_+$, *satisfying* $\mathbf{H}(\mathbf{s}, \varepsilon)$ *for some* $s \ge 2$, *such that* $|F_T(\alpha)| \le C \sup_{0 \le t \le T} \mathcal{V}(\alpha_t)$ *for every* $\alpha \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$; *furthermore,*

$$\sup_{n \ge 1} \nu^{(n)}(\omega, F_T) < +\infty \qquad a.s. \tag{28}$$

Proof. We prove (a) and (b) simultaneously. Let $\Upsilon^{(k)}$ be defined by $\Upsilon^{(k)} = F_T(\bar{X}^{(k)})$. We have

$$\begin{aligned}
&\frac{1}{H_n} \sum_{k=1}^{n} \eta_k \big(\Upsilon^{(k-1)} - \mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_{k-1}]\big) \\
&\quad = \frac{1}{H_n} \sum_{k=1}^{n} \eta_k \big(\Upsilon^{(k-1)} - \mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n]\big) && (29) \\
&\qquad + \frac{1}{H_n} \sum_{k=1}^{n} \eta_k \big(\mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n] - \mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_{k-1}]\big). && (30)
\end{aligned}$$

We have to prove that the right-hand side of (29) and (30) tend to 0 a.s. when $n \to +\infty$.

We first focus on the right-hand side of (29). From the very definition of $\tau(n,T)$, we have that $\{\bar{X}_t^{(k)}, 0 \le t \le T\}$ is $\bar{\mathcal{F}}_{\Gamma_n}$-measurable for $k \in \{0, \ldots, \tau(n,T)-1\}$. Hence, since F_T is $\sigma(\pi_s, 0 \le s \le T)$-measurable and $\bar{\mathcal{F}}_{\Gamma_n} \subset \mathcal{G}_n$, it follows that $\Upsilon^{(k)}$ is $\mathcal{G}_n$-measurable and that $\Upsilon^{(k)} = \mathbb{E}[\Upsilon^{(k)}/\mathcal{G}_n]$ for every $k \le \tau(n,T)-1$. Then, if F_T is bounded, we derive from $(\mathbf{C}_{\mathbf{3},\varepsilon})$ that

$$\begin{aligned}
\left| \frac{1}{H_n} \sum_{k=1}^{n} \eta_k \big(\Upsilon^{(k-1)} - \mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n]\big) \right| &\le \frac{2\|F_T\|_{\sup}}{H_n} \sum_{k=\tau(n,T)+1}^{n} \eta_k \le \frac{C}{H_n} \sum_{k=\tau(n,T)+1}^{n} \gamma_k H_k^{\varepsilon} \\
&\le \frac{C}{H_n^{1-\varepsilon}} (\Gamma_n - \Gamma_{\tau(n,T)})
\end{aligned}$$

$$\leq \frac{C(T)}{H_n^{1-\varepsilon}} \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.},$$

where we used the fact that $(H_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ are non-decreasing and non-increasing sequences, respectively.

Assume, now, that the assumptions of (b) are fulfilled with $\mathcal{V}$ satisfying $\mathbf{H}(\mathbf{s},\varepsilon)$ for some $s\geq 2$ and $\varepsilon<1$. By the Borel–Cantelli-like argument, it suffices to show that

$$\sum_{n\geq 1}\mathbb{E}\left[\left|\frac{1}{H_n^s}\sum_{k=\tau(n,T)+1}^{n}\eta_k(\Upsilon^{(k-1)}-\mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n])\right|^s\right]<+\infty. \tag{31}$$

Let us prove (31). Let $a_k:=\eta_k^{(s-1)/s}$ and $b_k(\omega):=\eta_k^{1/s}(\Upsilon^{(k-1)}-\mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n])$. The Hölder inequality applied with $\bar{p}=s/(s-1)$ and $\bar{q}=s$ yields

$$\left|\sum_{k=\tau(n,T)+1}^{n}a_kb_k(\omega)\right|^s\leq\left(\sum_{k=\tau(n,T)+1}^{n}\eta_k\right)^{s-1}\left(\sum_{k=\tau(n,T)+1}^{n}\eta_k|\Upsilon^{(k-1)}-\mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n]|^s\right).$$

Now, since $F_T(\alpha)\leq\sup_{0\leq t\leq T}\mathcal{V}(\alpha)$, it follows from the Markov property and from $\mathbf{H}(\mathbf{s},\varepsilon)$(i) that

$$\mathbb{E}[|F_T(\bar{X}^{(k)})|^s/\bar{\mathcal{F}}_{\Gamma_k}]\leq C\mathbb{E}\left[\sup_{0\leq t\leq T}\mathcal{V}^s(\bar{X}_t^{(k)})/\bar{\mathcal{F}}_{\Gamma_k}\right]\leq C_T\mathcal{V}^s(\bar{X}_{\Gamma_k}).$$

Then, using the two preceding inequalities and $(\mathbf{C_{3}}_{,\varepsilon})$ yields

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{k=\tau(n,T)+1}^{n}\eta_k(\Upsilon^{(k-1)}-\mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n])\right|^s\right]\\
&\leq C\left(\sum_{k=\tau(n,T)+1}^{n}\eta_k\right)^{s-1}\left(\sum_{k=\tau(n,T)+1}^{n}\eta_k\mathbb{E}[\mathcal{V}^s(\bar{X}_{\Gamma_{k-1}})]\right)\\
&\leq C\left(\sum_{k=\tau(n,T)+1}^{n}\eta_k\right)^{s}\mathbb{E}\left[\sup_{k=\tau(n,T)+1}^{n}\mathcal{V}^s(\bar{X}_{\Gamma_{k-1}})\right]\\
&\leq C\left(\sum_{k=\tau(n,T)+1}^{n}\gamma_kH_k^{\varepsilon}\right)^{s}\mathbb{E}\left[\sup_{t\in[0,S(n,T)]}\mathcal{V}^s(\bar{X}_t^{\tau(n,T)})\right],
\end{aligned}$$

where $S(n,T)=\Gamma_{n-1}-\Gamma_{\tau(n,T)}$ and C does not depend n. By the definition of $\tau(n,T)$, $S(n,T)\leq T$. Then, again using $\mathbf{H}(\mathbf{s},\varepsilon)$(i) yields

$$\sum_{n\geq 1}\mathbb{E}\left[\frac{1}{H_n^s}\left|\sum_{k=\tau(n,T)}^{n}\eta_k(\Upsilon^{(k-1)}-\mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n])\right|^s\right]\leq C\sum_{n\geq 1}\frac{1}{H_n^{s(1-\varepsilon)}}\mathbb{E}[\mathcal{V}^s(\bar{X}_{(\tau(n,T))})].$$

Since $n \mapsto N(n,T)$ is an increasing function, $n \mapsto \tau(n,T)$ is a non-decreasing function and $\mathrm{Card}\{n, \tau(n,T)=k\} = \Delta N(k+1,T) := N(k+1,T) - N(k,T)$. Then, since $n \mapsto H_n$ increases, a change of variable yields

$$\begin{aligned}
&\sum_{n\geq 1} \mathbb{E}\left[\frac{1}{H_n^s}\left|\sum_{k=\tau(n,T)+1}^{n} \eta_k(\Upsilon^{(k-1)} - \mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n])\right|^s\right] \\
&\quad \leq C \sum_{k\geq 1} \frac{\Delta N(k,T)}{H_k^{s(1-\varepsilon)}} \mathbb{E}[\mathcal{V}^s(\bar{X}_{\Gamma_{k-1}})] < +\infty,
\end{aligned}$$

by $\mathbf{H}(\mathbf{s},\varepsilon)$(iv).

Second, we prove that (30) tends to 0. For every $n \geq 1$, we let

$$M_n = \sum_{k=1}^{n} \frac{\eta_k}{H_k}(\mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_n] - \mathbb{E}[\Upsilon^{(k-1)}/\mathcal{G}_{k-1}]). \tag{32}$$

The process $(M_n)_{n\geq 1}$ is a $(\mathcal{G}_n)$-martingale and we want to prove that this process is L^2-bounded. Set $\Phi^{(k,n)} = \mathbb{E}[F_T(\bar{X}^{(k)})/\mathcal{G}_n] - \mathbb{E}[F_T(\bar{X}^{(k)})/\mathcal{G}_k]$. Since F_T is $\sigma(\pi_s, 0 \leq s \leq T)$-measurable, the random variable $\Phi^{(k,n)}$ is $\bar{\mathcal{F}}_{\Gamma_{N(k,T)}}$-measurable. Then, for every $i \in \{N(k,T), \ldots, n\}$, $\Phi^{(k,n)}$ is $\mathcal{G}_i$-measurable so that

$$\mathbb{E}[\Phi^{(i,n)}\Phi^{(k,n)}] = \mathbb{E}[\Phi^{(k,n)}\mathbb{E}[\Phi^{(i,n)}/\mathcal{G}_i]] = 0.$$

It follows that

$$\mathbb{E}[M_n^2] = \sum_{k\geq 1} \frac{\eta_k^2}{H_k^2}\mathbb{E}[(\Phi^{(k-1,n)})^2] + 2\sum_{k\geq 1}\frac{\eta_k}{H_k}\sum_{i=k+1}^{N(k-1,T)\wedge n}\frac{\eta_i}{H_i}\mathbb{E}[\Phi^{(i-1,n)}\Phi^{(k-1,n)}]. \tag{33}$$

Then,

$$\begin{aligned}
\sup_{n\geq 1}\mathbb{E}[M_n^2] &\leq \sum_{k\geq 1}\frac{\eta_k^2}{H_k^2}\sup_{n\geq 1}\mathbb{E}[(\Phi^{(k-1,n)})^2] + 2\sum_{k\geq 1}\frac{\eta_k}{H_k}\sum_{i=k+1}^{N(k-1,T)}\frac{\eta_i}{H_i}\sup_{n\geq 1}\mathbb{E}[\Phi^{(i-1,n)}\Phi^{(k-1,n)}] \\
&\leq C\Bigg(\sum_{k\geq 1}\frac{\eta_k}{H_k^{2-\varepsilon}}\sup_{n\geq 1}\mathbb{E}[(\Phi^{(k-1,n)})^2] \qquad (34)\\
&\qquad + \sum_{k\geq 1}\frac{\eta_k}{H_k^{2-\varepsilon}}\sum_{i=k+1}^{N(k-1,T)}\gamma_i\sup_{n\geq 1}\mathbb{E}[\Phi^{(i-1,n)}\Phi^{(k-1,n)}]\Bigg),
\end{aligned}$$

where, in the second inequality, we used assumption $(\mathbf{C_{3}}_{,\varepsilon})$ and the decrease of $i \mapsto 1/H_i^{1-\varepsilon}$. Hence, if F_T is bounded, using the fact that $\sum_{i=k+1}^{N(k-1,T)} \gamma_i \le T$ yields

$$\sup_{n\ge 1} \mathbb{E}[M_n^2] \le C \sum_{k\ge 1} \frac{\eta_k}{H_k^{2-\varepsilon}} \le C\left(\frac{\eta_1}{H_1^{2-\varepsilon}} + \int_{\eta_1}^{\infty} \frac{\mathrm{d}u}{u^{2-\varepsilon}}\right) < +\infty \tag{35}$$

since $\varepsilon < 1$. Assume, now, that the assumptions of (b) hold and let F_T be dominated by a function $\mathcal{V}$ satisfying $\mathbf{H}(\mathbf{s},\varepsilon)$. By the Markov property, the Jensen inequality and $\mathbf{H}(\mathbf{s},\varepsilon)$(i),

$$\mathbb{E}[(\Phi^{(k,n)})^2] \le C\mathbb{E}\left[\mathbb{E}\left[\sup_{0\le t\le T} \mathcal{V}^2(\bar{X}_t^{(k)})/\bar{\mathcal{F}}_{\Gamma_k}\right]\right] \le C_T \mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_k})].$$

We then derive from the Cauchy–Schwarz inequality that for every $n,k \ge 1$, for every $i \in \{k,\dots,N(k,T)\}$,

$$|\mathbb{E}[\Phi^{(i,n)}\Phi^{(k,n)}]| \le C\sqrt{\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_i})]}\sqrt{\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_k})]} \le C \sup_{t\in[0,T]} \mathbb{E}[\mathcal{V}^2(\bar{X}_t^{(k)})] \le C\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_k})],$$

where, in the last inequality, we once again used $\mathbf{H}(\mathbf{s},\varepsilon)$(i). It follows that

$$\sup_{n\ge 1} \mathbb{E}[M_n^2] \le C \sum_{k\ge 1} \frac{\eta_k}{H_k^{2-\varepsilon}} \mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_{k-1}})] < +\infty,$$

by $\mathbf{H}(\mathbf{s},\varepsilon)$(iii). Therefore, (34) is finite and (M_n) is bounded in L^2. Finally, we derive from the Kronecker lemma that

$$\frac{1}{H_n} \sum_{k=1}^{n} \eta_k (\mathbb{E}[F_T(\bar{X}^{(k-1)})/\mathcal{G}_n] - \mathbb{E}[F_T(\bar{X}^{(k-1)})/\mathcal{G}_{k-1}]) \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.}$$

As a consequence, $\sup_{n\ge 1} \nu^{(n)}(\omega, F_T) < +\infty$ a.s. if and only if

$$\sup_{n\ge 1} \frac{1}{H_n} \sum_{k=1}^{n} \mathbb{E}[F_T(\bar{X}^{(k-1)})/\mathcal{F}_{k-1}] < +\infty \qquad \text{a.s.}$$

This last property is easily derived from $\mathbf{H}(\mathbf{s},\varepsilon)$(i) and (ii). This completes the proof. □

Lemma 3. (a) *Assume* $(\mathbf{C_1})$ *and let* $x_0 \in \mathbb{R}^d$. *We then have* $\lim_{x\to x_0} \mathbb{E}[d(X^x, X^{x_0})] = 0$. *In particular, for every bounded Lispchitz (w.r.t. the distance* d*) functional* $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, *the function* Φ^F *defined by* $\Phi^F(x) = \mathbb{E}[F(X^x)]$ *is a (bounded) continuous function on* $\mathbb{R}^d$.

(b) *Assume* $(\mathbf{C_2})$. *For every compact set* $K \subset \mathbb{R}^d$,

$$\sup_{x\in K} \mathbb{E}[d(Y^{n,x}, X^x)] \overset{n\to+\infty}{\longrightarrow} 0. \tag{36}$$

Set $\Phi_n^F(x)=\mathbb{E}[F(Y^{n,x})]$. *Then, for every bounded Lispchitz functional* $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$,

$$\sup_{x\in K}|\Phi^F(x)-\Phi_n^F(x)|\overset{n\to+\infty}{\longrightarrow}0 \qquad \text{for every compact set } K\subset\mathbb{R}^d. \tag{37}$$

Proof. (a) By the definition of d, for every α, $\beta\in\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ and for every $T>0$,

$$d(\alpha,\beta)\le\left(1\wedge\sup_{0\le t\le T}|\alpha(t)-\beta(t)|\right)+\mathrm{e}^{-T}. \tag{38}$$

It easily follows from assumption $(\mathbf{C_1})$ and from the dominated convergence theorem that

$$\limsup_{x\to x_0}\mathbb{E}[d(X^x,X^{x_0})]\le\mathrm{e}^{-T}\qquad\text{for every } T>0.$$

Letting $T\to+\infty$ implies that $\lim_{x\to x_0}\mathbb{E}[d(X^x,X^{x_0})]=0$.

(b) We deduce from (38) and from assumption $(\mathbf{C_2})$ that for every compact set $K\subset\mathbb{R}^d$, for every $T>0$,

$$\limsup_{n\to+\infty}\sup_{x\in K}\mathbb{E}[d(Y^{n,x},X^x)]\le\mathrm{e}^{-T}.$$

Letting $T\to+\infty$ yields (36). □

Lemma 4. *Assume that* $(\eta_n)_{n\ge1}$ *and* (γ_n) *satisfy* $(\mathbf{C}_{\mathbf{3},\varepsilon})$ *with* $\varepsilon<1$ *and (14). Then:*

(i) *for every* $t\ge0$, *for every bounded continuous function* $f:\mathbb{R}^d\to\mathbb{R}$,

$$\nu_t^{(n)}(\omega,f)-\nu_0^{(n)}(\omega,f)\overset{n\to+\infty}{\longrightarrow}0\qquad a.s.;$$

(ii) *if, moreover, a.s., every weak limit* $\nu^{(\infty)}(\omega,\mathrm{d}\alpha)$ *of* $(\nu^{(n)}(\omega,\mathrm{d}\alpha))_{n\ge1}$ *is the distribution of a Markov process with semigroup* $(Q_t^\omega)_{t\ge0}$, *then, a.s.,* $\nu^{(\infty)}(\omega,\mathrm{d}\alpha)$ *is the distribution of a stationary process.*

Proof. (i) Let $f:\mathbb{R}^d\to\mathbb{R}$ be a bounded continuous function. Since $\bar{X}_t^{(k)}=\bar{X}_{\Gamma_{N(k,t)}}$, we have

$$\nu_t^{(n)}(\omega,f)-\nu_0^{(n)}(\omega,f)=\frac{1}{H_n}\sum_{k=1}^n\eta_k(f(\bar{X}_{\Gamma_{N(k-1,t)}})-f(\bar{X}_{\Gamma_{k-1}})).$$

From the very definition of $N(n,T)$ and $\tau(n,T)$, one checks that $N(k-1,T)\le n-1$ if and only if $\tau(n,T)\ge k$. Then,

$$\frac{1}{H_n}\sum_{k=1}^n\eta_kf(\bar{X}_{\Gamma_{k-1}})=\frac{1}{H_n}\sum_{k=1}^{\tau(n,t)}\eta_{N(k-1,t)+1}f(\bar{X}_{\Gamma_{N(k-1,t)}})$$
$$+\frac{1}{H_n}\sum_{k=1}^n\eta_kf(\bar{X}_{\Gamma_{k-1}})1_{\{k-1\notin N(\{0,\ldots,n\},t)\}}.$$

It follows that

$$\nu_t^{(n)}(\omega, f) - \nu_0^{(n)}(\omega, f) = \frac{1}{H_n}\sum_{k=1}^{\tau(n,t)} (\eta_k - \eta_{N(k-1,t)+1}) f(\bar{X}_{\Gamma_{N(k-1,t)}})$$

$$+ \frac{1}{H_n}\sum_{\tau(n,t)+1}^{n} \eta_k f(\bar{X}_{\Gamma_{N(k-1,t)}})$$

$$- \frac{1}{H_n}\sum_{k=1}^{n} \eta_k f(\bar{X}_{\Gamma_{k-1}}) 1_{\{k-1 \notin N(\{0,\ldots,n\},t)\}}.$$

Then, since f is bounded and since

$$\sum_{k=1}^{n} \eta_k 1_{\{k-1 \notin N(\{0,\ldots,n\},t)\}} = \sum_{k=1}^{n} \eta_k - \sum_{k=1}^{\tau(n,t)} \eta_{N(k-1,t)+1}$$

$$\leq \sum_{k=1}^{\tau(n,t)} |\eta_k - \eta_{N(k-1,t)+1}| + \sum_{k=\tau(n,t)+1}^{n} \eta_k,$$

we deduce that

$$|\nu_t^{(n)}(\omega, f) - \nu_0^{(n)}(\omega, f)| \leq 2\|f\|_\infty \left(\frac{1}{H_n}\sum_{k=1}^{\tau(n,t)} |\eta_k - \eta_{N(k-1,t)+1}| + \frac{1}{H_n}\sum_{k=\tau(n,t)+1}^{n} \eta_k \right).$$

Hence, we have to show that the sequences of the right-hand side of the preceding inequality tend to 0. On the one hand, we observe that

$$|\eta_k - \eta_{N(k-1,t)+1}| \leq \sum_{\ell=k+1}^{N(k-1,T)+1} |\eta_\ell - \eta_{\ell-1}| \leq \max_{\ell \geq k+1} \frac{|\Delta \eta_\ell|}{\gamma_\ell} \sum_{\ell=k}^{N(k-1,T)+1} \gamma_\ell.$$

Using the fact that $\sum_{\ell=k}^{N(k-1,T)+1} \gamma_\ell \leq T + \gamma_1$ and condition (14) yields

$$\frac{1}{H_n}\sum_{k=1}^{\tau(n,t)} |\eta_k - \eta_{N(k-1,t)+1}| \overset{n\to+\infty}{\longrightarrow} 0.$$

On the other hand, by $(\mathbf{C_{3,\varepsilon}})$, we have

$$\frac{1}{H_n}\sum_{k=\tau(n,T)+1}^{n} \eta_k \leq \frac{C}{H_n^{1-\varepsilon}} \sum_{k=\tau(n,T)+1}^{n} \gamma_k \leq \frac{CT}{H_n^{1-\varepsilon}} \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.},$$

which completes the proof of (i).

(ii) Let $\mathbb{Q}_+$ denote the set of non-negative rational numbers. Let $(f_\ell)_{\ell\geq 1}$ be an everywhere dense sequence in $\mathcal{C}_K(\mathbb{R}^d)$ endowed with the topology of uniform convergence on compact sets. Since $\mathbb{Q}_+$ and $(f_\ell)_{\ell\geq 1}$ are countable, we derive from (i) that there exists $\tilde{\Omega}\subset\Omega$ such that $\mathbb{P}(\tilde{\Omega})=1$ and such that for every $\omega\in\tilde{\Omega}$, every $t\in\mathbb{Q}_+$ and every $\ell\geq 1$,

$$\nu_t^{(n)}(\omega,f_\ell)-\nu_0^{(n)}(\omega,f_\ell)\overset{n\to+\infty}{\longrightarrow}0.$$

Let $\omega\in\tilde{\Omega}$ and let $\nu^{(\infty)}(\omega,\mathrm{d}\alpha)$ denote a weak limit of $(\nu^{(n)}(\omega,\mathrm{d}\alpha))_{n\geq 1}$. We have

$$\nu_t^{(\infty)}(\omega,f_\ell)=\nu_0^{(\infty)}(\omega,f_\ell)\qquad\forall t\in\mathbb{Q}_+\ \forall\ell\geq 1$$

and we easily deduce that

$$\nu_t^{(\infty)}(\omega,f)=\nu_0^{(\infty)}(\omega,f)\qquad\forall t\in\mathbb{R}_+\ \forall f\in\mathcal{C}_K(\mathbb{R}^d).$$

Hence, if $\nu^{(\infty)}(\omega,\mathrm{d}\alpha)$ is the distribution of a Markov process (Y_t) with semigroup $(Q_t^\omega)_{t\geq 0}$, we have, for all $f\in\mathcal{C}_K(\mathbb{R}^d)$,

$$\int Q_t^\omega f(x)\nu_0^{(\infty)}(\omega,\mathrm{d}x)=\int f(x)\nu_0^{(\infty)}(\omega,\mathrm{d}x)\qquad\forall t\geq 0.$$

$\nu_0^{(\infty)}(\omega,\mathrm{d}x)$ is then an invariant distribution for (Y_t). This completes the proof. □

4.2. Proof of Theorem 1

Thanks to Lemma 1(a) applied with $E=\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ and d defined by (6),

$$\nu^{(n)}(\omega,\mathrm{d}\alpha)\overset{(Sk)}{\Longrightarrow}\mathbb{P}_{\nu_0}(\mathrm{d}\alpha)\qquad\text{a.s.}\iff\nu^{(n)}(\omega,F)\overset{n\to+\infty}{\longrightarrow}\int F(x)\mathbb{P}_{\nu_0}(\mathrm{d}x)\qquad\text{a.s.}\tag{39}$$

for every bounded Lipschitz functional $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$. Now, consider such a functional. By the assumptions of Theorem 1, we know that a.s., $(\nu_0^{(n)}(\omega,\mathrm{d}x))_{n\geq 1}$ converges weakly to ν_0. Set $\Phi^F(x):=\mathbb{E}[F(X^x)]$, $x\in\mathbb{R}^d$. By Lemma 3(a), Φ^F is a bounded continuous function on $\mathbb{R}^d$. It then follows from $(\mathbf{C_{0,1}})$ that

$$\frac{1}{H_n}\sum_{k=1}^n\eta_k\Phi^F(\bar{X}_0^{(k-1)})\overset{n\to+\infty}{\longrightarrow}\int\Phi^F(x)\nu_0(\mathrm{d}x)=\int F(x)\mathbb{P}_{\nu_0}(\mathrm{d}x)\qquad\text{a.s.}$$

Hence, the right-hand side of (39) holds for F as soon as

$$\frac{1}{H_n}\sum_{k=1}^n\eta_k(F(\bar{X}^{(k-1)})-\Phi^F(\bar{X}_0^{(k-1)}))\overset{n\to+\infty}{\longrightarrow}0\qquad\text{a.s.}\tag{40}$$

Let us prove (40). First, let $T>0$ and let F_T be defined by (7). By Lemma 2,

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k F_T(\bar{X}^{(k-1)}) - \frac{1}{H_n}\sum_{k=1}^{n}\eta_k \mathbb{E}[F_T(\bar{X}^{(k-1)})/\bar{\mathcal{F}}_{\Gamma_{k-1}}] \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.} \tag{41}$$

With the notation of Lemma 3(b), we derive from assumption $(\mathbf{C_2})$(i) that

$$\mathbb{E}[F_T(\bar{X}^{(k-1)})/\bar{\mathcal{F}}_{\Gamma_{k-1}}] = \Phi_k^{F_T}(\bar{X}_0^{(k-1)}).$$

Let $N\in\mathbb{N}$. On one hand, by Lemma 3(b),

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k(\Phi_k^{F_T}(\bar{X}_0^{(k-1)}) - \Phi^{F_T}(\bar{X}_0^{(k-1)}))1_{\{|\bar{X}_0^{(k-1)}|\le N\}} \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.} \tag{42}$$

On the other hand, the tightness of $(\nu_0^{(n)}(\omega,\mathrm{d}x))_{n\ge1}$ on $\mathbb{R}^d$ yields

$$\psi(\omega,N) := \sup_{n\ge1}(\nu_0^{(n)}(\omega,(B(0,N)^c))) \overset{N\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.}$$

It follows that, a.s.,

$$\begin{aligned}&\sup_{n\ge1}\left(\frac{1}{H_n}\sum_{k=1}^{n}\eta_k|\Phi_k^{F_T}(\bar{X}_0^{(k-1)}) - \Phi^{F_T}(\bar{X}_0^{(k-1)})|1_{\{|\bar{X}_0^{(k-1)}|>N\}}\right)\\ &\quad\le 2\|F\|_\infty\psi(\omega,N) \overset{N\to+\infty}{\longrightarrow} 0.\end{aligned} \tag{43}$$

Hence, a combination of (42) and (43) yields

$$\forall T>0 \qquad \frac{1}{H_n}\sum_{k=1}^{n}\eta_k(\Phi_k^{F_T}(\bar{X}_0^{(k-1)}) - \Phi^{F_T}(\bar{X}_0^{(k-1)})) \overset{n\to+\infty}{\longrightarrow} 0 \qquad \text{a.s.} \tag{44}$$

Finally, let $(T_\ell)_{\ell\ge1}$ be a sequence of positive numbers such that, $T_\ell\to+\infty$ when $\ell\to+\infty$. Combining (44) and (41), we obtain that, a.s., for every $\ell\ge1$,

$$\begin{aligned}&\limsup_{n\to+\infty}\left|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k(F(\bar{X}^{(k-1)}) - \Phi^F(\bar{X}^{(k-1)}))\right|\\ &\quad\le \limsup_{n\to+\infty}\left|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k(F(\bar{X}^{(k-1)}) - F_{T_\ell}(\bar{X}^{(k-1)}))\right|\\ &\qquad + \limsup_{n\to+\infty}\left|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k(\Phi^{F_{T_\ell}}(\bar{X}_0^{(k-1)}) - \Phi^F(\bar{X}_0^{(k-1)}))\right|.\end{aligned}$$

By the definition of d, $|F - F_{T_\ell}| \le \mathrm{e}^{-T_\ell}$. Then, a.s.,

$$\limsup_{n\to+\infty}\left|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(F(\bar{X}^{(k-1)}) - \Phi^F(\bar{X}_0^{(k-1)})\big)\right| \le 2\mathrm{e}^{-T_\ell} \qquad \forall \ell \ge 1.$$

Letting $\ell \to +\infty$ implies (40).

The generalization to non-bounded functionals in Theorem 1 is then derived from (28) and from a uniform integrability argument.

4.3. Proof of Theorem 2

(i) We want to prove that the conditions of Lemma 1(b) are fulfilled. Since $(\nu_0^{(n)}(\omega, \mathrm{d}x))_{n\ge1}$ is supposed to be a.s. tight, one can check that for every bounded Lipschitz functional $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, (40) is still valid. Then, let $(F_\ell)_{\ell\ge1}$ be a sequence of bounded Lipschitz functionals. There exists $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for every $\omega \in \tilde{\Omega}$, $(\nu_0^{(n)}(\omega, \mathrm{d}x))_{n\ge1}$ is tight and

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(F_\ell(\bar{X}^{(k-1)}(\omega)) - \Phi^{F_\ell}(\bar{X}_0^{(k-1)}(\omega))\big) \overset{n\to+\infty}{\longrightarrow} 0 \qquad \forall \ell \ge 1. \tag{45}$$

Let $\omega \in \tilde{\Omega}$ and let $\phi_\omega : \mathbb{N} \mapsto \mathbb{N}$ be an increasing function. As $(\nu_0^{(\phi_\omega(n))}(\omega, \mathrm{d}x))_{n\ge1}$ is tight, there exists a convergent subsequence $(\nu_0^{(\phi_\omega\circ\psi_\omega(n))}(\omega, \mathrm{d}x))_{n\ge1}$. We denote its weak limit by ν_∞. Since Φ^{F_ℓ} is continuous for every $\ell \ge 1$ (see Lemma 3(a)),

$$\nu_0^{(\phi_\omega\circ\psi_\omega(n))}(\omega, \Phi^{F_\ell}) \overset{n\to+\infty}{\longrightarrow} \nu_\infty(\Phi^{F_\ell}) = \int F_\ell(\alpha)\mathbb{P}_{\nu_\infty}(\mathrm{d}\alpha) \qquad \forall \ell \ge 1.$$

We then derive from (45) that for every $\ell \ge 1$

$$\nu^{(\phi_\omega\circ\psi_\omega(n))}(\omega, F_\ell) \overset{n\to+\infty}{\longrightarrow} \int F_\ell(\alpha)\mathbb{P}_{\nu_\infty}(\mathrm{d}\alpha).$$

It follows that the conditions of Lemma 1(b) are fulfilled with $\mathcal{U} = \{\mathbb{P}_\mu, \mu \in \mathcal{I}\}$, where

$$\mathcal{I} = \left\{\mu \in \mathcal{P}(\mathbb{R}^d), \exists \omega \in \tilde{\Omega} \text{ and an increasing function } \phi : \mathbb{N} \mapsto \mathbb{N}, \mu = \lim_{n\to+\infty} \nu^{(\phi(n))}(\omega, \mathrm{d}\alpha)\right\}.$$

Hence, by Lemma 1(b), we deduce that $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\ge1}$ is a.s. tight with $\mathcal{U}$-valued limits.

Finally, Theorem 2(ii) is a consequence of condition (14) and Lemma 4(ii).

5. Path-dependent option pricing in stationary stochastic volatility models

In this section, we propose a simple and efficient method to price options in stationary stochastic volatility (SSV) models. In most stochastic volatility (SV) models, the volatility is a mean reverting process. These processes are generally ergodic with a unique invariant distribution (the Heston model or the BNS model for instance (see below) but also the SABR model (see Hagan *et al.* [8]), ...). However, they are usually considered in SV models under a non-stationary regime, starting from a deterministic value (which usually turns out to be the mean of their invariant distribution). However, the instantaneous volatility is not easy to observe on the market since it is not a traded asset. Hence, it seems to be more natural to assume that it evolves under its stationary regime than to give it a deterministic value at time 0.[4]

From a purely calibration viewpoint, considering an SV model in its SSV regime will not modify the set of parameters used to generate the implied volatility surface, although it will modify its shape, mainly for short maturities. This effect can in fact be an asset of the SSV approach since it may correct some observed drawbacks of some models (see, e.g., the Heston model below).

From a numerical point of view, considering SSV models is no longer an obstacle, especially when considering multi-asset models (in the unidimensional case, the stationary distribution can be made more or less explicit like in the Heston model; see below) since our algorithm is precisely devised to compute by simulation some expectations of functionals of processes under their stationary regime, even if this stationary regime cannot be directly simulated.

As a first illustration (and a benchmark) of the method, we will describe in detail the algorithm for the pricing of Asian options in a Heston model. We will then show in our numerical results to what extent it differs, in terms of smile and skew, from the usual SV Heston model for short maturities. Finally, we will complete this section with a numerical test on Asian options in the BNS model where the volatility is driven by a tempered stable subordinator. Let us also mention that this method can be applied to other fields of finance like interest rates, and commodities and energy derivatives where mean-reverting processes play an important role.

[4]When one has sufficiently close observations of the stock price, it is in fact possible to derive a rough idea of the size of the volatility from the variations of the stock price (see, e.g., Jacod [10]). Then, using this information, a good compromise between a deterministic initial value and the stationary case may be to assume that the distribution μ_0 of the volatility at time 0 is concentrated around the estimated value (see Section 2.2 for application of our algorithm in this case).

5.1. Option pricing in the Heston SSV model

We consider a Heston stochastic volatility model. The dynamic of the asset price process $(S_t)_{t\geq 0}$ is given by $S_0 = s_0$ and

$$\begin{aligned} \mathrm{d}S_t &= S_t(r\,\mathrm{d}t + \sqrt{(1-\rho^2)v_t}\,\mathrm{d}W_t^1 + \rho\sqrt{v_t}\,\mathrm{d}W_t^2), \\ \mathrm{d}v_t &= k(\theta - v_t)\,\mathrm{d}t + \varsigma\sqrt{v_t}\,\mathrm{d}W_t^2, \end{aligned}$$

where r denotes the interest rate, (W^1, W^2) is a standard two-dimensional Brownian motion, $\rho \in [-1, 1]$ and k, θ and ς are some non-negative numbers. This model was introduced by Heston in 1993 (see Heston [9]). The equation for (v_t) has a unique (strong) pathwise continuous solution living in $\mathbb{R}_+$. If, moreover, $2k\theta > \varsigma^2$, then (v_t) is a positive process (see Lamberton and Lapeyre [11]). In this case, (v_t) has a unique invariant probability ν_0. Moreover, $\nu_0 = \gamma(a, b)$ with $a = (2k)/\varsigma^2$ and $b = (2k\theta)/\varsigma^2$. In the following, we will assume that (v_t) is in its stationary regime, that is, that

$$\mathcal{L}(v_0) = \nu_0.$$

5.1.1. *Option price and stationary processes*

Using our procedure to price options in this model naturally needs to express the option price as the expectation of a functional of a stationary stochastic process.

Naïve method. (may work) Since $(v_t)_{t\geq 0}$ is stationary, the first idea is to express the option price as the expectation of a functional of $(v_t)_{t\geq 0}$: by Itô calculus, we have

$$S_t = s_0 \exp\left(\left(rt - \frac{1}{2}\int_0^t v_s\,\mathrm{d}s\right) + \rho\int_0^t \sqrt{v_s}\,\mathrm{d}W_s^2 + \sqrt{1-\rho^2}\int_0^t \sqrt{v_s}\,\mathrm{d}W_s^1\right). \tag{46}$$

Since

$$\int_0^t \sqrt{v_s}\,\mathrm{d}W_s^2 = \Lambda(t, (v_t)) := \frac{v_t - v_0 - k\theta t + k\int_0^t v_s\,\mathrm{d}s}{\varsigma},$$

it follows by setting $M_t = \int_0^t \sqrt{v_s}\,\mathrm{d}W_s^1$ that

$$S_t = \Psi(t, (v_s), (M_s)), \tag{47}$$

where Ψ is given for every $t \geq 0$, u and $w \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ by

$$\Psi(t, u, w) = s_0 \exp\left(\left(rt - \frac{1}{2}\int_0^t u(s)\,\mathrm{d}s\right) + \rho\Lambda(t, u) + \sqrt{1-\rho^2}w(t)\right).$$

Then, let $F : \mathcal{C}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ be a non-negative measurable functional. Conditioning by $\mathcal{F}_T^{W^2}$ yields

$$\mathbb{E}[F_T((S_t)_{t\geq 0})] = \mathbb{E}[\tilde{F}_T((v_t)_{t\geq 0})],$$

where, for every $u \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$,

$$\tilde{F}_T(u) = \mathbb{E}\left[F_T\left(\left(\Psi\left(t, u, \int_0^t \sqrt{u(s)}\,\mathrm{d}W_s^1\right)\right)_{t\geq 0}\right)\right].$$

For some particular options such as the European call or put (thanks to the Black–Scholes formula), the functional $\tilde{F}$ is explicit. In those cases, this method seems to be very efficient (see Panloup [20] for numerical results). However, in the general case, the computation of $\tilde{F}$ will need some Monte Carlo methods at each step. This approach is then very time-consuming in general – that is why we are going to introduce another representation of the option as a functional of a stationary process.

General method. (always works) We express the option premium as the expectation of a functional of a two-dimensional stationary stochastic process. This method is based on the following idea. Even though (v_t, M_t) is not stationary, (S_t) can be expressed as a functional of a stationary process (v_t, y_t). Indeed, consider the following SDE given by

$$\begin{cases} \mathrm{d}y_t = -y_t\,\mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t^1, \\ \mathrm{d}v_t = k(\theta - v_t)\,\mathrm{d}t + \varsigma\sqrt{v_t}\,\mathrm{d}W_t^2. \end{cases} \tag{48}$$

First, one checks that the SDE has a unique strong solution and that assumption $(\mathbf{S_1})$ is fulfilled with $V(x_1, x_2) = 1 + x_1^2 + x_2^2$. This ensures the existence of an invariant distribution $\tilde{\nu}_0$ for the SDE (see, e.g., Pagès [17]). Then, since (v_t) is positive and has a unique invariant distribution, the uniqueness of the invariant distribution follows. Then, assume that $\mathcal{L}(y_0, v_0) = \tilde{\nu}_0$. Since $(v_t, M_t) = (v_t, y_t - y_0 + \int_0^t y_s\,\mathrm{d}s)$, we have, for every positive measurable functional $F : \mathcal{C}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[F_T((S_t)_{t\geq 0})] &= \mathbb{E}[F_T((\psi(t, v_t, M_t))_{t\geq 0})] \\ &= \mathbb{E}_{\tilde{\nu_0}}\left[F_T\left(\left(\psi\left(t, v_t, y_t - y_0 + \int_0^t y_s\,\mathrm{d}s\right)\right)_{t\geq 0}\right)\right], \end{aligned} \tag{49}$$

where $\mathbb{P}_{\tilde{\nu_0}}$ is the stationary distribution of the process (v_t, y_t). Every option price can then be expressed as the expectation of an explicit functional of a stationary process. We will develop this second general approach in the numerical tests below.

Remark 9. The idea of the second method holds for every stochastic volatility model for which (S_t) can be written as follows:

$$S_t = \Phi\left(t, v_t, \sum_{i=1}^{p} \int_0^t h_i(|v_s|)\,\mathrm{d}Y_s^i\right), \tag{50}$$

where, for every $i \in \{1, \dots, p\}$, $h_i : \mathbb{R}_+ \to \mathbb{R}$ is a positive function such that $h_i(x) = o(|x|)$ as $|x| \to +\infty$, (Y_t^i) is a square-integrable centered Lévy process and (v_t) is a mean reverting stochastic process solution to a Lévy driven SDE.

In some complex models, showing the uniqueness of the invariant distribution may be difficult. In fact, it is important to note at this stage that the uniqueness of the invariant distribution for the couple (v_t, y_t) is not required. Indeed, by construction, the local martingale (M_t) does not depend on the choice of y_0. It follows that if $\mathcal{L}(y_0, v_0) = \tilde{\mu}$, with $\tilde{\mu}$ constructed such that $\mathcal{L}(v_0) = \nu_0$, (49) still holds. This implies that it is only necessary that uniqueness holds for the invariant distribution of the stochastic volatility process.

5.1.2. Numerical tests on Asian options

We recall that (v_t) is a Cox–Ingersoll–Ross process. For this type of processes, it is well known that the genuine Euler scheme cannot be implemented since it does not preserve the non-negativity of the (v_t). That is why some specific discretization schemes have been studied by several authors (Alfonsi [1], Deelstra and Delbaen [5] and Berkaoui *et al.* [4, 6]). In this paper, we consider the scheme studied by the last authors in a decreasing step framework. We denote it by $(\bar{v}_t)$. We set $\bar{v}_0 = x > 0$ and

$$\bar{v}_{\Gamma_{n+1}} = |\bar{v}_{\Gamma_n} + k\gamma_{n+1}(\theta - \bar{v}_{\Gamma_n}) + \varsigma\sqrt{\bar{v}_{\Gamma_n}}(W^2_{\Gamma_{n+1}} - W^2_{\Gamma_n})|.$$

We also introduce the stepwise constant Euler scheme $(\bar{y}_t)$ of $(y_t)_{t\geq 0}$ defined by

$$\bar{y}_{\Gamma_{n+1}} = \bar{y}_{\Gamma_n} - \gamma_{n+1}\bar{y}_{\Gamma_n} + \sqrt{\bar{v}_{\Gamma_n}}(\tilde{W}^1_{\Gamma_{n+1}} - \tilde{W}^1_{\Gamma_n}), \qquad \bar{y}_0 = y \in \mathbb{R}^d.$$

Denote by $(\bar{v}^{(k)}_t)$ and $(\bar{y}^{(k)}_t)$ the shifted processes defined by $\bar{v}^{(k)}_t := \bar{v}_{\Gamma_k + t}$ and $\bar{y}^{(k)}_t = \bar{y}_{\Gamma_k + t}$, and let $(\nu^{(n)}(\omega, \mathrm{d}\alpha))_{n\geq 1}$ be the sequence of empirical measures defined by

$$\nu^{(n)}(\omega, \mathrm{d}\alpha) = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k 1_{\{(\bar{v}^{(k-1)}, \bar{y}^{(k-1)}) \in d\alpha\}}.$$

The specificity of both the model and the Euler scheme implies that Theorems 1 and 2 cannot be directly applied here. However, a specific study using the fact that (9) holds for every compact set of $\mathbb{R}^*_+ \times \mathbb{R}$ when $2k\theta/\varsigma^2 > 1 + 2\sqrt{6}/\varsigma$ (see Theorem 2.2 of Berkaoui *et al.* [4] and Remark 9) shows that

$$\nu^{(n)}(\omega, \mathrm{d}\alpha) \overset{n\to+\infty}{\Longrightarrow} \mathbb{P}_{\tilde{\nu}_0}(\mathrm{d}\alpha) \qquad \text{a.s.}$$

when $2k\theta/\varsigma^2 > 1 + 2\sqrt{6}/\varsigma$. Details are left to the reader.

Let us now state our numerical results obtained for the pricing of Asian options with this discretization. We denote by $C_{as}(\nu_0, K, T)$ and $P_{as}(\nu_0, K, T)$ the Asian call and put prices in the SSV Heston model. We have

$$C_{as}(\nu_0, K, T) = \mathrm{e}^{-rT}\mathbb{E}_{\nu_0}\left[\left(\frac{1}{T}\int_0^T S_s\,\mathrm{d}s - K\right)_+\right]$$

and

$$P_{as}(\nu_0,K,T)=\mathrm{e}^{-rT}\mathbb{E}_{\nu_0}\left[\left(K-\frac{1}{T}\int_0^T S_s\,\mathrm{d}s\right)_+\right].$$

With the notation of (49), approximating $C_{as}(\nu_0,K,T)$ and $P_{as}(\nu_0,K,T)$ by our procedure needs to simulate the sequences $(C_{as}^n)_{n\geq 1}$ and $(P_{as}^n)_{n\geq 1}$ defined by

$$C_{as}^n=\frac{1}{H_n}\sum_{k=1}^n \eta_k \mathrm{e}^{-rT}\left(\frac{1}{T}\int_0^T \Psi(s,\bar{v}^{(k-1)},\bar{M}^{(k-1)})\,\mathrm{d}s-K\right)_+,$$

$$P_{as}^n=\frac{1}{H_n}\sum_{k=1}^n \eta_k \mathrm{e}^{-rT}\left(K-\frac{1}{T}\int_0^T \Psi(s,\bar{v}^{(k-1)},\bar{M}^{(k-1)})\,\mathrm{d}s\right)_+.$$

These sequences can be computed by the method developed in Section 1.3. Note that the specific properties of the exponential function and the linearity of the integral imply that $(\int_0^T \Psi(t,\bar{v}^{(n-1)},\bar{M}^{(n-1)})\,\mathrm{d}s)$ can be computed quasi-recursively.

Let us state our numerical results for the Asian call with parameters

$$\begin{aligned} &s_0=50, \qquad r=0.05, \qquad T=1, \qquad \rho=0.5,\\ &\theta=0.01, \qquad \varsigma=0.1, \qquad k=2. \end{aligned} \tag{51}$$

We also assume that $K\in\{44,\ldots,56\}$ and choose the following steps and weights: $\gamma_n=\eta_n=n^{-1/3}$. In Table 1, we first state the reference value for the Asian call price obtained for $N=10^8$ iterations. In the two following lines, we state our results for $N=5.10^4$ and $N=5.10^5$ iterations. Then, in the last lines, we present the numerical results obtained

Table 1. Approximation of the Asian call price

K	44	45	46	47	48	49	50
Asian call (ref.)	**6.92**	**5.97**	**5.04**	**4.12**	**3.25**	**2.46**	**1.78**
$N=5\cdot 10^4$	6.89	6.07	5.07	4.13	3.18	2.49	1.77
$N=5\cdot 10^5$	6.90	6.02	5.00	4.11	3.24	2.46	1.79
$N=5\cdot 10^4$ (CP parity)	6.92	5.96	5.04	4.13	3.26	2.46	1.78
$N=5\cdot 10^5$ (CP parity)	6.92	5.97	5.04	4.12	3.25	2.47	1.78

K	51	52	53	54	55	56
Asian call (ref.)	**1.23**	**0.82**	**0.53**	**0.33**	**0.21**	**0.12**
$N=5\cdot 10^4$	1.21	0.81	0.51	0.34	0.22	0.11
$N=5\cdot 10^5$	1.23	0.82	0.53	0.33	0.21	0.13
$N=5\cdot 10^4$ (CP parity)	1.23	0.82	0.53	0.31	0.21	0.12
$N=5\cdot 10^5$(CP parity)	1.23	0.82	0.53	0.33	0.21	0.13

using the call-put parity

$$C_{as}(\nu_0, K, T) - P_{as}(\nu_0, S_0, K, T) = \frac{s_0}{rT}(1 - \mathrm{e}^{-rT}) - K\mathrm{e}^{-rT} \tag{52}$$

as a means of variance reduction. The computation times for $N = 5.10^4$ and $N = 5.10^5$ (using MATLAB with a Xeon 2.4 GHz processor) are about 5 s and 51 s, respectively. In particular, the complexity is quasi-linear and the additional computations needed when we use the call-put parity are negligible.

5.2. Implied volatility surfaces of Heston SSV and SV models

Given a particular pricing model (with initial value s_0 and interest rate r) and its associated European call prices denoted by $C_{\mathrm{eur}}(K,T)$, we recall that the implied volatility surface is the graph of the function $(K,T) \mapsto \sigma_{\mathrm{imp}}(K,T)$, where $\sigma_{\mathrm{imp}}(K,T)$ is defined for every maturity $T > 0$ and strike K as the unique solution of

$$C_{BS}(s_0, K, T, r, \sigma_{\mathrm{imp}}(K,T)) = C_{\mathrm{eur}}(K,T),$$

where $C_{BS}(s_0, K, T, r, \sigma)$ is the price of the European call in the Black–Scholes model with parameters s_0, r and σ. When $C_{\mathrm{eur}}(K,T)$ is known, the value of $\sigma_{\mathrm{imp}}(K,T)$ can be numerically computed using the Newton method or by dichotomy if the first method is not convergent.

In this last part, we compare the implied volatility surfaces induced by the SSV and SV Heston models where we suppose that the initial value of (v_t) in the SV Heston model is the mean of the invariant distribution, that is, we suppose that $v_0 = \theta$.[5] We also assume that the parameters are those of (51), except the correlation coefficient ρ.

In Figures 1 and 2, the volatility curves obtained when $T = 1$ are depicted, whereas in Figures 3 and 4, we set the strike K at $K = 50$ and let the time vary. These representations show that when the maturity is long, the differences between the SSV and SV Heston models vanish. This is a consequence of the convergence of the stochastic volatility to its stationary regime when $T \to +\infty$.

The main differences between these models then appear for short maturities. That is why we complete this part by a representation of the volatility curve when $T = 0.1$ for $\rho = 0$ and $\rho = 0.5$ in Figures 5 and 6, respectively. We observe that for short maturities, the volatility smile is more curved and the skew is steeper. These phenomena seem interesting for calibration since one well-known drawback of the standard Heston model is that it can have overly flat volatility curves for short maturities.

5.3. Numerical tests on Asian options in the BNS SSV model

The BNS model introduced in Barndorff-Nielsen and Shephard [3] is a stochastic volatility model where the volatility process is a Lévy-driven positive Ornstein–Uhlenbeck process.

[5]This choice is the most usual in practice.

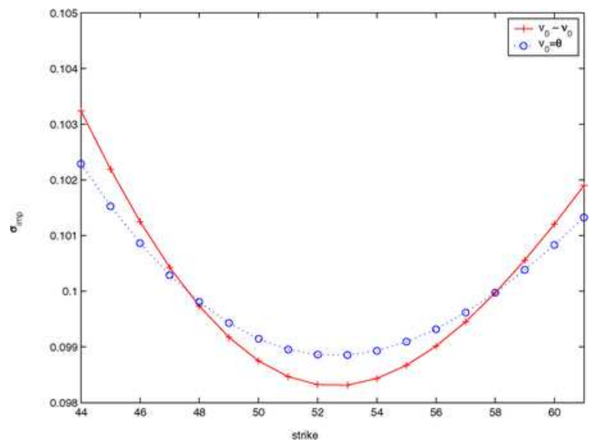

Figure 1. $\rho = 0,\ K \mapsto \sigma_{\text{imp}}(K,1)$.

The dynamic of the asset price (S_t) is given by $S_t = S_0 \exp(X_t)$,

$$
\begin{aligned}
\mathrm{d}X_t &= (r - \tfrac{1}{2}v_t)\,\mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t + \rho\,\mathrm{d}Z_t, \qquad \rho \le 0,\\
\mathrm{d}v_t &= -\mu v_t\,\mathrm{d}t + \mathrm{d}Z_t, \qquad \mu > 0,
\end{aligned}
$$

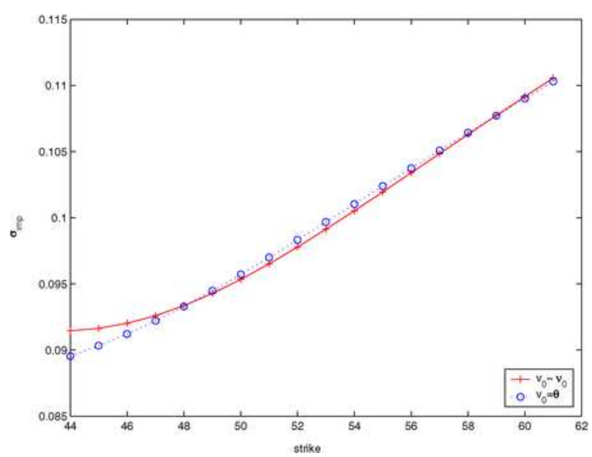

Figure 2. $\rho = 0.5,\ K \mapsto \sigma_{\text{imp}}(K,1)$.

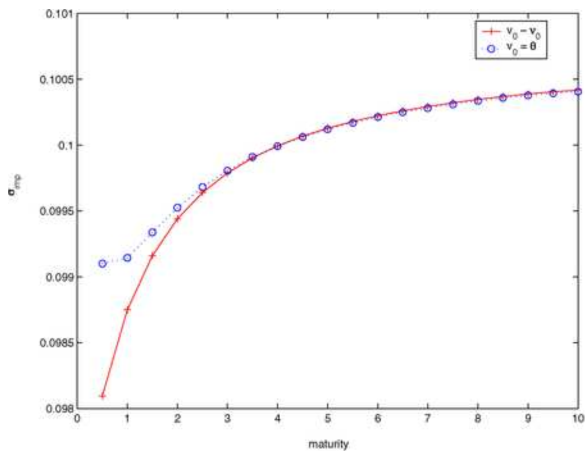

Figure 3. $\rho = 0$, $T \mapsto \sigma_{\text{imp}}(50, T)$.

where (Z_t) is a subordinator without drift term and Lévy measure π. In the following, we assume that (Z_t) is a tempered stable subordinator, that is, that

$$\pi(\mathrm{d}y) = 1_{\{y>0\}} \frac{c \exp(-\lambda y)}{y^{1+\alpha}} \, \mathrm{d}y, \qquad c > 0, \lambda > 0, \alpha \in (0, 1).$$

As in the Heston model, we want to use our algorithm as a way of option pricing when the stochastic volatility evolves under its stationary regime and test it on Asian options using the method described in detail in Section 5.1. This model does not require a specific

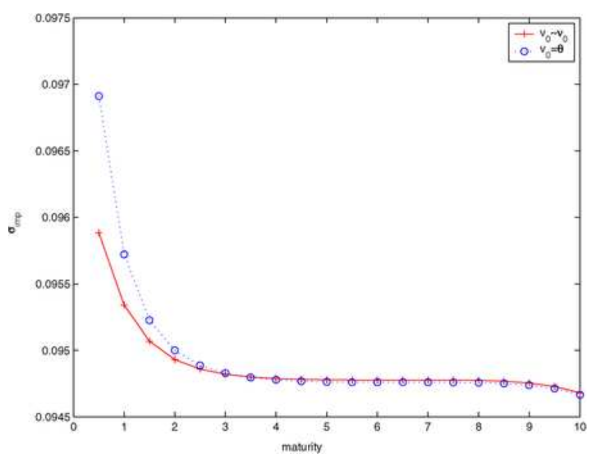

Figure 4. $\rho = 0.5$, $T \mapsto \sigma_{\text{imp}}(50, T)$.

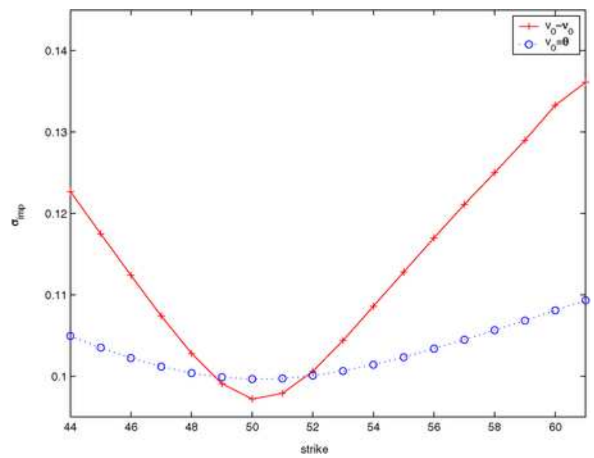

Figure 5. $\rho = 0$, $T \mapsto \sigma_{\mathrm{imp}}(50, T)$.

discretization and the approximate Euler scheme (P) (see Section 3.2) relative to (v_t) can be implemented using the rejection method. In Table 2, we present our numerical results obtained for the following choices of parameters, steps and weights:

$$\rho = -1, \qquad \lambda = \mu = 1, \qquad c = 0.01, \qquad \alpha = \tfrac{1}{2}, \qquad \gamma_n = \eta_n = n^{-1/3}.$$

The computation times for $N = 5.10^4$ and $N = 5.10^5$ are about 8.5 s and 93 s, respectively. Note that for this model, the convergence seems to be slower because of the approximation of the jump component.

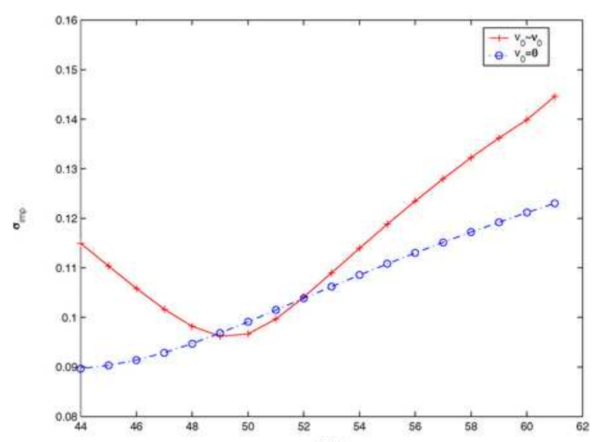

Figure 6. $\rho = 0.5$, $T \mapsto \sigma_{\mathrm{imp}}(50, T)$.

Table 2. Approximation of the Asian call price in the BNS model

K	44	45	46	47	48	49	50
Asian call (ref.)	**6.75**	**5.83**	**4.93**	**4.05**	**3.18**	**2.35**	**1.57**
$N = 5 \cdot 10^4$	6.83	5.91	5.01	4.10	3.22	2.35	1.51
$N = 5 \cdot 10^5$	6.78	5.86	4.96	4.06	3.19	2.34	1.52
$N = 5 \cdot 10^4$ (CP parity)	6.76	5.85	4.94	4.07	3.20	2.29	1.51
$N = 5 \cdot 10^5$ (CP parity)	6.75	5.83	4.93	4.04	3.17	2.32	1.54

K	51	52	53	54	55	56
Asian call (ref.)	**0.91**	**0.55**	**0.39**	**0.29**	**0.23**	**0.18**
$N = 5 \cdot 10^4$	0.77	0.46	0.33	0.27	0.22	0.19
$N = 5 \cdot 10^5$	0.79	0.48	0.34	0.27	0.21	0.17
$N = 5 \cdot 10^4$ (CP parity)	0.79	0.47	0.37	0.27	0.23	0.19
$N = 5 \cdot 10^5$(CP parity)	0.83	0.50	0.36	0.28	0.22	0.17

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